

# VOLUMES OF $\mathrm{SL}_n\mathbb{C}$ -REPRESENTATIONS OF FUNDAMENTAL GROUPS OF HYPERBOLIC 3-MANIFOLD GROUPS

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**ABSTRACT.** Let  $M$  be a compact oriented three-manifold whose interior is hyperbolic of finite volume. We prove a variation formula for the volume on the variety of representations of  $M$  in  $\mathrm{SL}_n(\mathbb{C})$ . Our proof follows the strategy of Reznikov's rigidity when  $M$  is closed, in particular we use Fuks' approach to variations by means of Lie algebra cohomology. When  $n = 2$ , we get back Hodgson's formula for variation of volume on the space of hyperbolic Dehn fillings. Our formula also yields the variation of volume on the space of decorated triangulations obtained by [1] and [10].

## 1. INTRODUCTION

Let  $M$  be a compact oriented three-manifold whose interior admits a complete hyperbolic metric of finite volume. There is a well defined notion of volume of a representation of its fundamental group  $\pi_1 M$  in  $\mathrm{SL}_n(\mathbb{C})$ , see Definition 5 for instance. We view the volume as a function on the variety of representations  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))$ . Bucher-Burger-Iozzi [6] have shown that the volume is maximal precisely at the composition of the lifts of the holonomy with the irreducible representation  $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_n(\mathbb{C})$ . If  $M$  is furthermore closed, then this volume function is constant on connected components of  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))$  (see [20]) but in the non-compact case the volume can vary locally. When  $n = 2$  this variety of representations (up to conjugation) contains the space of hyperbolic structures on the manifold, and the volume has been intensively studied in this case; in particular, a variation formula was obtained in Hodgson's thesis [15, Chapter 5], by means of Schläfli's variation formula for polygons in hyperbolic space. The variation of the volume was also discussed in [1, 13] when  $n = 3$ , and in [10] through the study of decorated triangulations of hyperbolic manifolds.

The purpose of this paper is to produce a formula for the variation of volume in  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))$  for arbitrary  $n$  and for differentiable deformations of any representation, independently of the existence of decorated triangulations. The variety of representations has deformations that are nontrivial up to conjugation; more precisely the component of  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))/\mathrm{SL}_n(\mathbb{C})$  that contains the representation of maximal volume has dimension  $n - 1$  [17]. Our results are proved in  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))$ , but they apply with no change to  $\mathrm{Hom}(\pi_1 M, \mathrm{PSL}_n(\mathbb{C}))$ .

The boundary of  $M$  consists of  $k \geq 1$  tori,  $T_1^2, \dots, T_k^2$ . Fix the orientation of  $\partial M$  corresponding to the outer normal, as in Stokes theorem, and choose  $l_i, m_i$  ordered generators of  $\pi_1(T_i^2)$ , so that if we view them as oriented curves, they generate the induced orientation. For instance, for the exterior of an oriented knot in  $S^3$ , we can take  $l_1$  as a longitude and  $m_1$  as a meridian, with  $l_1$  following the orientation of the knot and  $m_1$  as describing the positive sense of rotation. For a complex number  $z \in \mathbb{C}$ , denote by  $\Re(z)$  and  $\Im(z)$  its real and imaginary parts respectively. Assume now  $\rho_t$  is a path of representations in  $\mathrm{Hom}(\pi_1 M, \mathrm{SL}_n(\mathbb{C}))$  parametrized by  $t \in I \subset \mathbb{R}$ . As a consequence of the Lie-Kolchin Theorem, there exist 1-parameter families of matrices  $A_i(t) \in \mathrm{SL}_n(\mathbb{C})$  and of upper

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2010 *Mathematics Subject Classification.* 57M50 14D20 (57R20, 57T10).

*Key words and phrases.* Volume, hyperbolic manifold, characteristic class, representation variety, Schläfli formula, flat bundle.

First author was supported by Mineco grant MTM2016-80439-P.

Second author was supported by Mineco grant MTM2015-66165-P .

triangular matrices  $a_i(t), b_i(t) \in \mathfrak{sl}_n(\mathbb{C})$  so that

$$(1) \quad \rho_t(l_i) = A_i(t) \exp(a_i(t)) A_i(t)^{-1} \quad \text{and} \quad \rho_t(m_i) = A_i(t) \exp(b_i(t)) A_i(t)^{-1}.$$

Our main result states:

**Theorem 1.** *Assume that  $A_i(t)$ ,  $a_i(t)$ , and  $b_i(t)$  as in (1) are differentiable. Then the volume is differentiable and*

$$\frac{d}{dt} \text{vol}(M, \rho_t) = \sum_{i=1}^k \text{tr}(\Re(b_i) \Im(\dot{a}_i) - \Re(a_i) \Im(\dot{b}_i)).$$

For  $n = 2$  this formula is precisely Hodgson's (see Subsection 6.2), and for  $n = 3$  it is equivalent to the variation on the space of decorated triangulations by Bergeron, Falbel and Guilloux [1, 13], for  $n = 3$ , and by Dimofte, Gabela and Goncharov for general  $n$  [10].

Our argument is a generalization of Reznikov's proof of the rigidity of the volume for closed manifolds [20]. At the heart of Reznikov's argument is the fact that the volume of a representation  $\rho$  can be seen as a characteristic class of the horizontal foliation on the total space of the flat principal bundle on  $M$  induced by  $\rho$ . This characteristic class comes from a cohomology class of the Lie algebra  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ , i.e. it is induced by a class in  $H^3(\mathfrak{g})$ . The study of the variation of this characteristic class then relies on results by Fuks in [11], he shows in particular that the variation of volume itself can be interpreted as a characteristic class of a foliation and this class stems from a cohomology class in  $H^2(\mathfrak{g}; \mathfrak{g}^\vee)$ , where  $\mathfrak{g}^\vee$  is the dual Lie algebra, viewed as a  $\mathfrak{g}$ -module. But this cohomology group is trivial, as follows by a classical result of Cartier [9], see Corollary 2, hence the volume for  $M$  compact is locally constant. We aim to follow the same outline in the non-closed case, which technically amounts to extend the homological tools used by Reznikov to a relative setting. This is the plan of this work:

Firstly, in Section 2 we develop the homological tools needed for our construction: we give a definition of cohomology groups of an object relatively to a family of subobjects. As it is difficult to find a single place in the literature where all the relative versions of the maps we need are explained, we start by defining in a unified way the relative cohomology constructions we will use, this is inspired by the work of Bieri-Eckmann [2] on relative cohomology of groups, but with a stronger emphasis on the pair object-subobject. The relative cohomology groups are devised in such a way that, by definition, if  $G$  is an object and  $\{A\}$  is a family of subobjects, then the cohomology of  $G$  relative to  $\{A\}$  fits into a long exact sequence:

$$\cdots \longrightarrow H^n(G) \longrightarrow \prod H^n(A) \longrightarrow H^n(G; \{A\}) \longrightarrow H^{n+1}(G) \longrightarrow \cdots$$

We also discuss the relations between our definitions and previous existing notions of relative cohomology groups.

Secondly, in Section 3 we use the relative cohomological tools of the previous section to give relative versions of the constructions of Fuks [11] on characteristic classes of foliations and variations of those. This gives the conceptual framework in which we can state and prove our formula. Up to this point we work in a general context, so as to pave the way for future applications.

In the compact case the volume of a representation  $\rho : \pi_1(M) \rightarrow \text{SL}_n(\mathbb{C})$  is defined as a pull-back of a universal hyperbolic volume class in the continuous cohomology group  $H_c^3(\text{SL}_n(\mathbb{C}))$ ; since the peripheral subgroups of a non-compact finite volume hyperbolic manifold are all abelian, the cohomology group where we want to look for a universal relative volume class is  $H_c^3(\text{SL}_n(\mathbb{C}); \{B\})$ : the continuous cohomology groups of  $\text{SL}_n(\mathbb{C})$  relative to the family  $\{B\}$  consisting of its Borel subgroups. This program for constructing the volume is carried out and explained in Section 4 where we also show that the definition through relative cohomology corresponds to the common definitions in literature, for instance the one given in [6] via the use of the transfer map in continuous-bounded cohomology. The key point for our construction is the crucial fact that

continuous-bounded cohomology of an amenable group is trivial, hence we have a canonical isomorphism  $H_{cb}^3(\mathrm{SL}_n(\mathbb{C}); \{B\}) \rightarrow H_{cb}^3(\mathrm{SL}_n(\mathbb{C}))$  that allows to interpret the classical universal hyperbolic volume cohomology class as a relative cohomology class.

The study of the variation of the volume requires us then to find explicit cocycle representatives for the relative volume cohomology class. This is the object of Section 5. The main ingredient in this part of our work is the fact, underlying the Van Est isomorphism connecting the continuous cohomology of a real connected Lie group  $G$  with maximal compact subgroup  $K$  and the cohomology of its Lie algebra, that the equivariant de Rham complex of the symmetric space  $G/K$ ,  $\Omega_{dR}^*(G/K)^G$ , computes the continuous cohomology of  $G$ ; our cocycle will then appear as a bounded differential 3-form on  $G/K$  with a specific choice of trivialization on each Borel subgroup. Here we also show how to express the value and its variation as a characteristic class on the total space of the flat bundle induced by a representation.

Finally in Section 6 we collect our efforts and prove our variation formula and give some consequences.

**Acknowledgements :** We would like to thank Julien Marché for helpful conversations on this subject.

## 2. RELATIVE COHOMOLOGY

Our approach to define relative cohomology relies on the following three crucial points:

- (1) The existence of functorial chain complexes that compute the cohomology groups we want to relativize.
- (2) The fact that given a family of objects  $(A_i)_{i \in I}$  and coefficients  $V_i$  and functorial chain complexes  $C^*(A_i; V_i)$ , the product chain complex  $\prod_{i \in I} C^*(A_i; V_i)$  has as  $n$ -cohomology the product of cohomologies  $\prod_{i \in I} H^n(A_i; V_i)$ .
- (3) The fact that the cone of a chain map between chain complexes is functorial in the homotopy category of complexes of  $\mathbb{R}$ -vector spaces.

**2.1. The cone construction.** Consider two cochain complexes of  $\mathbb{R}$ -vector spaces, i.e. differentials rise degree by one, and a chain map  $f : K^* \rightarrow L^*$ . By definition  $\mathrm{Cone}(f)^*$ , the cone of  $f$ , is the cochain complex given by:

$$\mathrm{Cone}(f)^n = L^{n-1} \oplus K^n \text{ and } d_{\mathrm{Cone}(f)} = \begin{pmatrix} -d_L & f \\ 0 & d_K \end{pmatrix}.$$

where  $d_{\mathrm{Cone}(f)}$  acts on column vectors.

One checks that, as expected in any reasonable definition of a *relative* cocycle, an element  $\begin{pmatrix} l \\ k \end{pmatrix} \in L^{n-1} \oplus K^n$  is an  $n$ -cocycle if and only if  $k$  is a cocycle in  $K^n$  and  $d_L(l) = f^n(k)$ . For such a pair we will call  $k$  the *absolute* part and  $l$  the *relative* part.

This construction is functorial in the following sense. If we have a commutative square of maps of chain complexes:

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ r \downarrow & & \downarrow s \\ A & \xrightarrow{g} & B \end{array}$$

then we have an induced chain map:  $\mathrm{Cone}(r, s) : \mathrm{Cone}(f)^* \rightarrow \mathrm{Cone}(g)^*$ , given by

$$\mathrm{Cone}(r, s) = \begin{pmatrix} s & 0 \\ 0 & r \end{pmatrix}$$

The main use of  $\text{Cone}(f)^*$  is that its homology interpolates between that of  $L$  and that of  $K$ ; indeed by construction there is a short exact sequence of complexes:

$$0 \rightarrow L[-1] \rightarrow \text{Cone}(f) \rightarrow K \rightarrow 0,$$

where  $L[-1]$  is the shifted complex  $L[-1]^n = L^{n-1}$ ,  $d_{L[-1]} = -d_L$ . This sequence splits in each degree and by standard techniques gives rise to a long exact sequence in cohomology.

$$\cdots \longrightarrow H^{n-1}(L) \longrightarrow H^n(\text{Cone}(f)) \longrightarrow H^n(K) \xrightarrow{\delta} H^n(L) \longrightarrow \cdots$$

One checks directly by unwinding the definitions that the connecting homomorphism  $\delta : H^*(K) \rightarrow H^*(L)$  coincides with  $H^*(f)$ . As expected, if we are given a morphism  $(r, s)$  between maps of chain complexes, then we will have an induced commuting ladder in cohomology:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^{n-1}(L) & \longrightarrow & H^n(\text{Cone}(f)) & \longrightarrow & H^n(K) \xrightarrow{\delta} H^n(L) \longrightarrow \cdots \\ & & \downarrow H^{n-1}(s) & & \downarrow H^n(\text{Cone}(r,s)) & & \downarrow H^n(r) \\ \cdots & \longrightarrow & H^{n-1}(B) & \longrightarrow & H^n(\text{Cone}(g)) & \longrightarrow & H^n(A) \xrightarrow{\delta} H^n(B) \longrightarrow \cdots \end{array}$$

## 2.2. Relative cohomology.

**Definition 1.** Let  $H^*$  be our cohomology theory, possibly with coefficients (e.g. discrete group cohomology). If the cohomology theory admits coefficients we assume that the functorial chain complexes computing the cohomology with coefficients are functorial in both variables.

Let  $G$  be an object (Lie algebra, Lie group, manifold etc.) and  $(A_i)_{i \in I}$  a family of subobjects, possibly with repetitions. If the theory admits coefficients we consider also a coefficient  $V$  for the object  $G$ , coefficients  $W_i$  for each object  $A_i$  and maps between coefficients compatible with the inclusions  $A_i \hookrightarrow G$ , so that we have an induced map for each  $i \in I$ :  $C^*(G; V) \rightarrow C^*(A_i; W_i)$

Then we define the relative cohomology of  $G$  with coefficients in  $V$  with respect to  $\{A_i\}, \{W_i\}$  and we denote by  $H^*(G, \{A_i\}; V, \{W_i\})$ , the cohomology of the cone of the canonical map  $C^*(G, V) \rightarrow \prod_{i \in I} C^*(A_i, W_i)$ .

As usual, if both  $V$  and the  $W_i$ 's are trivial coefficients, then we simply write  $H^*(G, \{A_i\})$  for the relative cohomology group. Concretely, a relative  $n$ -cocycle in  $C^n(G, \{A_i\}; V, \{W_i\})$  is a pair  $(c, \{a_i\}_{i \in I})$  where  $c$  is an ordinary  $n$ -cocycle for  $G$  with coefficients in  $V$  which is a coboundary (i.e. trivial) on each subobject  $A_i$  when the coefficients are restricted to  $W_i$ , together with a specific trivialization  $a_i$  on each subobject  $A_i$ .

The following properties of the relative cohomology groups are immediate from the functoriality of the cochain complexes  $C^*(G; V)$  and  $C^*(A_i; W_i)$  and that of the cone construction:

**Proposition 1.** (1) The relative cohomology groups  $H^*(G, \{A_i\}; V, \{W_i\})$  are functorial in both pairs  $(G, (A_i)_{i \in I})$  and  $(V, \{W_i\})$ .

(2) The relative cohomology groups fit into a long exact sequence:

$$\begin{aligned} \cdots \longrightarrow \prod_{i \in I} H^{n-1}(A_i; W_i) &\xrightarrow{\delta} H^n(G, \{A_i\}; V, \{W_i\}) \longrightarrow H^n(G; V) \\ &\longrightarrow \prod_{i \in I} H^n(A_i; W_i) \longrightarrow \cdots \end{aligned}$$

(3) If  $J \subset I$  is a subset of the indexing family for the subobjects  $A_i$ , then we have an induced natural transformation in relative cohomology

$$H^*(G, \{A_i\}_{i \in I}; V, \{W_i\}_{i \in I}) \rightarrow H^*(G, \{A_i\}_{i \in J}; V, \{W_i\}_{i \in J}).$$

**2.3. Examples.** The different objects and cohomologies we have in mind are:

- (1) Continuous or smooth cohomology of a Lie group. Here  $G$  is a Lie group, for our purposes  $\mathrm{SL}_n(\mathbb{C})$ , and each  $A_i$  a closed subgroup, for us a Borel subgroup of  $G$ . We take for  $C^*(G; \mathbb{R})$  the continuous or smooth *normalized* bar resolution  $C_c^*(G; \mathbb{R})$  or  $C_\infty^*(G; \mathbb{R})$  [4, Chap. IX]. In this case, by a classical result of Hoschild-Mostow, the canonical inclusion map  $C_\infty^*(G; \mathbb{R}) \rightarrow C_c^*(G; \mathbb{R})$  is a quasi-isomorphism.  
Another functorial way to compute the cohomology underlies Van Est theorem (see Section 5.2). Given a semi-simple Lie group with associated symmetric space  $G/K$ , then the subcomplex of the de Rham complex of  $G$ -invariant differential forms computes the continuous cohomology of  $G$ :  $H^*(\Omega(G/K)^K) \simeq H_c^*(G; \mathbb{R})$ . This resolution is functorial in the category of pairs semi-simple Lie group – maximal compact subgroup.
- (2) Continuous-bounded cohomology. Since the only case we are interested in is for Lie groups with trivial coefficients we may use the cochain complex of continuous-bounded functions  $C_{cb}^*(G; \mathbb{R})$ .
- (3) Cohomology of discrete groups. Here  $G$  is a discrete group,  $A_i$  a family of subgroups and  $C^*(G; \mathbb{R})$  stands for the usual bar resolution. This can of course be viewed as a particular case of continuous cohomology.
- (4) De Rham cohomology of manifolds. In this case  $G$  is a smooth manifold,  $A_i$  a family of smooth submanifolds, typically the connected components of the boundary, and  $C^*(G; \mathbb{R}) = \Omega_{dR}^*(G)$  is the de Rham complex of smooth differential forms on  $M$ .
- (5) Lie group cohomology [21, Chap. 7]. Here  $G$  and  $A$  are respectively a real Lie algebra and a family of Lie subalgebras. For  $C^*(G; \mathbb{R})$  we use the so-called *standard resolution* of Chevalley-Eilenberg. It is only in this case that we will need to consider non trivial coefficients.

For some of these theories one can find in the literature other relative cohomology theories, and the one here presented coincides with these except for one important case: Lie algebra cohomology. Let us review briefly this.

**Relative cohomology for discrete groups:** This has been defined by Bieri-Eckmann in [2]. Their construction defines the relative cohomology  $H^*(G, \{A_i\})$  as the absolute cohomology of the group  $G$  with coefficients in a specific non-trivial  $G$ -module. As they explain in [2, p. 282], their construction is isomorphic to ours, up to a sign in the long exact sequence of the pair  $(G, \{A_i\}_{i \in I})$ . Fortunately for us this gives in our case a reformulation of their geometric interpretation of relative group cohomology without sign problems:

**Theorem 2.** [2, Thm 1.3] *Let  $(X, Y)$  be an Eilenberg-MacLane pair  $K(G, \{A_i\}; 1)$ . Then the relative cohomology sequences of  $X$  modulo  $Y$  and of  $G$  modulo  $\{A_i\}_{i \in I}$  are isomorphic. More precisely one has a commuting ladder with vertical isomorphisms:*

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & H^n(G, \{A_i\}) & \longrightarrow & H_{dR}^n(G) & \longrightarrow & H^{n+1}(A_i) & \longrightarrow & H^{n+1}(G, \{A_i\}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & H^n(X, Y) & \longrightarrow & H^n(X) & \longrightarrow & H^{n+1}(A) & \longrightarrow & H^{n+1}(M, A) & \longrightarrow & \cdots
 \end{array}$$

Where the cohomology in the bottom is the usual long exact sequence in singular cohomology.

We will be particularly interested in the case where  $X = M$  is an hyperbolic manifold of finite volume, and  $A = \partial M$  is its boundary, in which case  $A$  is a finite disjoint union of tori, i.e.  $K(\mathbb{Z}^2, 1)$ 's.

**De Rham cohomology:** Given a manifold  $M$  and a smooth submanifold  $A$ , a usual way to define relative cohomology groups  $H_{dR}^*(M, A)$  is to consider the kernel  $\Omega_{dR}^*(M, A)$  of the canonical map between de Rham complexes induced by the inclusion:  $\Omega_{dR}^*(M) \rightarrow \Omega_{dR}^*(A)$ . This gives rise to a

level-wise split short exact sequence of complexes:

$$0 \longrightarrow \Omega_{dR}^*(M, A) \longrightarrow \Omega_{dR}^*(M) \longrightarrow \Omega_{dR}^*(A) \longrightarrow 0$$

where the surjectivity uses the tubular neighborhood to extend any differential form on  $A$  to a form on  $M$ . As these are chain complexes of  $\mathbb{R}$ -vector spaces, the usual argument based on the snake lemma gives rise to a long exact sequence:

$$\cdots \longrightarrow H_{dR}^n(M, A) \longrightarrow H_{dR}^n(M) \longrightarrow H_{dR}^{n+1}(A) \longrightarrow H_{dR}^{n+1}(M, A) \longrightarrow \cdots$$

The relative de Rham can also be defined using the cone construction, cf. [5]. There is a canonical map

$$\Omega_{dR}^*(M, A) \rightarrow \text{Cone}(\Omega_{dR}^*(M) \rightarrow \Omega_{dR}^*(A)),$$

it maps a differential form  $\omega$  of degree  $n$  that is zero on  $A$  to  $(0, \omega) \in \Omega_{dR}^{n-1}(A) \oplus \Omega_{dR}^n(M) = \Omega_{dR}(M, \{A\})$ . It is immediate to check that this is a map of chain complexes, compatible with the restriction map and the connecting homomorphisms, and hence gives a commutative ladder;

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{dR}^n(M, A) & \longrightarrow & H_{dR}^n(M) & \longrightarrow & H_{dR}^n(A) \longrightarrow H_{dR}^{n+1}(M, A) \longrightarrow \cdots \\ & & \downarrow & & \parallel & & \parallel & & \downarrow \\ \cdots & \longrightarrow & H_{dR}^n(M, \{A\}) & \longrightarrow & H_{dR}^n(M) & \longrightarrow & H_{dR}^n(A) \longrightarrow H_{dR}^{n+1}(M, \{A\}) \longrightarrow \cdots \end{array}$$

where in the bottom we denote by  $H_{dR}^*(M, \{A\})$  "our" relative cohomology groups. Applying the five lemma we conclude that this canonical map is a quasi-isomorphism.

**Lie algebra cohomology:** There is an important case where our relative groups do not coincide with the usual ones: Lie algebra cohomology. Given a Lie algebra  $\mathfrak{g}$  and a subalgebra  $\mathfrak{h}$ , Chevalley-Eilenberg [21] define the relative Lie algebra cohomology via the (now known as) relative Chevalley-Eilenberg complex:

$$H^*(\mathfrak{g}, \mathfrak{h}) = H^*(\text{Hom}_{\mathfrak{h}\text{-mod}}(\bigwedge^* \mathfrak{g}/\mathfrak{h}, \mathbb{R})).$$

If  $\mathfrak{g}$  and  $\mathfrak{h}$  are Lie algebras of a Lie group  $G$  and a closed subgroup  $H$ , the relationship between the cohomologies  $H^*(\mathfrak{g})$ ,  $H^*(\mathfrak{h})$  and  $H^*(\mathfrak{g}, \mathfrak{h})$  parallels the relationship between the cohomologies of the spaces in the fibration sequence:

$$H \rightarrow G \rightarrow G/H.$$

In particular there is a Hochschild-Serre spectral sequence relating these cohomologies, in contrast with the long exact sequence in our case.

To distinguish our definition and to avoid an unnecessary clash with standard notations, even in case we have a family of subobjects consisting of a single element, we will denote our relative version as  $H^*(\mathfrak{g}, \{\mathfrak{h}\})$  and keep the usual notation  $H^*(\mathfrak{g}, \mathfrak{h})$  for the cohomology of the complex  $\text{Hom}_{\mathfrak{h}\text{-mod}}(\bigwedge^* \mathfrak{g}/\mathfrak{h}, \mathbb{R})$ .

**Continuous-bounded cohomology:** Continuous-bounded cohomology produces cohomology groups that are naturally Banach spaces, and this is an important feature of the theory. As we will have to consider non-countable families of subgroups, there is no hope that we could give some metric to our relative cohomology groups  $H_{cb}^*(G, \{A_i\})$  in any way compatible with the metrics on the absolute cohomology groups, for the space  $\prod_{i \in I} H_{cb}^*(A_i)$  will not usually be metrizable. However we are only interested in these relative cohomology groups as tools interpolating between the cohomology of a group and the cohomologies of subgroups in a given family and we will not enter the subtler point of the metric.

**Notation 1.** As a general rule we will write cohomology with coefficients separated by semicolons, eg.  $H^3(\text{SL}_n(\mathbb{C}); \mathbb{C}^n)$ , unless we are dealing with trivial coefficients, in which case we will usually omit them, and write  $H_c^3(\text{SL}_n(\mathbb{C}))$  instead of  $H_c^3(\text{SL}_n(\mathbb{C}); \mathbb{R})$ . For cochain complexes we will however keep the reference even to trivial coefficients.

## 3. RELATIVE CHARACTERISTIC AND VARIATION MAPS

In this section we explain how one can “relativize” Fuks construction [11, Chap. 3, Par. 1] of a characteristic class of a foliation, and more generally of a manifold with  $\mathfrak{g}$ -structure, and the way he handles their variation.

**3.1. Relative characteristic classes.** Given a manifold  $E$  and a flat connection  $\nabla \in \Omega_{dR}^1(E, \mathfrak{g})$  on  $E$  with values in a Lie algebra  $\mathfrak{g}$ , the absolute characteristic class map is given on the cochain level by:

$$\begin{aligned} \text{Char}_\nabla : C^*(\mathfrak{g}; \mathbb{R}) &\longrightarrow \Omega_{dR}^*(E) \\ \alpha &\longmapsto (X_1, \dots, X_n) \rightsquigarrow \alpha(\nabla X_1, \dots, \nabla X_n). \end{aligned}$$

This construction is contravariantly functorial in both variables  $\mathfrak{g}$  and  $E$ ; flatness of  $\nabla$  implies this is in fact a chain map, i.e. it commutes with the differentials.

Fix a family of Lie subalgebras  $\{\mathfrak{b}\}$  of  $\mathfrak{g}$  and a family of smooth closed submanifolds  $\{A\} \subset E$ , for instance the family of connected components of the boundary of  $E$ . Assume that the flat connection  $\nabla$  on  $A$  restricts to a flat connection with values in  $\mathfrak{b}_A$ , an element in the chosen family of Lie subalgebras. Then by functoriality of the map  $\text{Char}_\nabla$  we have for each  $A \subset E$  a commutative diagram:

$$\begin{array}{ccc} C^*(\mathfrak{g}; \mathbb{R}) & \xrightarrow{\text{Char}_\nabla} & \Omega_{dR}^*(E) \\ \downarrow & & \downarrow \\ C^*(\mathfrak{b}_A; \mathbb{R}) & \xrightarrow{\text{Char}_{\nabla|_A}} & \Omega_{dR}^*(A) \end{array}$$

By functoriality of the cone construction we get a *relative* characteristic cochain map:

$$C^*(\mathfrak{g}, \{\mathfrak{b}\}) \xrightarrow{\text{Char}_\nabla, \{\nabla|_A\}} \Omega_{dR}^*(E, \{A\})$$

that induces the “characteristic classes” map in cohomology.

**3.2. Variation of characteristic classes.** Let us again briefly recall Fuks framework in the absolute case [11, Chap. 3, pp. 241-246]. We consider a 1-parameter family of flat connections  $\nabla_t$  on a manifold  $E$  with values in a *fixed* Lie algebra  $\mathfrak{g}$ . Given a Lie algebra cohomology class  $[\omega] \in H^*(\mathfrak{g}; \mathbb{R})$ , we want to understand the variation of the cohomology class  $\text{Char}_{\nabla_t}(\omega) \in H_{dR}^*(E)$  as  $t$  varies.

Assume that the connection  $\nabla_t$  depends smoothly on  $t$ , then its derivative in  $t = 0$ , denoted by  $\dot{\nabla}_0$ , is again a connection with values in  $\mathfrak{g}$ . The characteristic class  $\text{Char}_{\nabla_t}(\alpha)$  depends then also smoothly on the parameter  $t$  and, assuming  $\alpha$  is of degree  $n$ , its derivative at  $t = 0$  is directly computed to be the de Rham cohomology class of the form obtained by the Leibniz derivative rule:

$$\text{Var}_{\nabla_t}(\alpha) : (X_1, \dots, X_n) \mapsto \sum_{i=1}^n \alpha(\nabla_0 X_1, \dots, \nabla_0 X_{i-1}, \dot{\nabla}_0 X_i, \nabla_0 X_{i+1}, \dots, \nabla_0 X_n).$$

From this we get a chain map:

$$\begin{aligned} \text{Var}_{\nabla_t} : C^*(\mathfrak{g}; \mathbb{R}) &\longrightarrow \Omega_{dR}^*(E) \\ \alpha &\longmapsto \text{Var}_{\nabla_t}(\alpha). \end{aligned}$$

The family of connections  $\nabla_t$  can also be seen as a single connection on  $E$  but with values in the algebra of currents

$$\widetilde{\mathfrak{g}} = \mathcal{C}^\infty(\mathbb{R}, \mathfrak{g}).$$

The associated characteristic class map

$$\text{Char}_{\nabla_t} : H^*(\widetilde{\mathfrak{g}}) \rightarrow H_{dR}^*(E)$$

factors the variation map in a very nice way. Consider the following two cochain maps:

(1) The map  $\text{var}$ :

$$\begin{aligned} \text{var}: C^m(\mathfrak{g}; \mathbb{R}) &\longrightarrow C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee) \\ \alpha &\longmapsto (g_1, \dots, g_{n-1}) \rightsquigarrow (h \mapsto \alpha(g_1, \dots, g_{n-1}, h)) \end{aligned}$$

where by  $\mathfrak{g}^\vee$  denotes the dual vector space  $\text{Hom}(\mathfrak{g}, \mathbb{R})$ , this is canonically a left  $\mathfrak{g}$ -module given by  $(g\phi)(h) = -\phi([g, h])$ .

(2) Fuks map [11, Chap. 3 p. 244] is a cochain map, in fact a split monomorphism:

$$F: C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee) \longrightarrow C^n(\widetilde{\mathfrak{g}})$$

that sends a cochain  $\alpha \in C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee)$  to the cochain

$$(\phi_1, \dots, \phi_n) \mapsto \sum_{i=1}^n (-1)^{n-i} \left[ \alpha(\phi_1(0), \dots, \phi_{i-1}(0), \widehat{\phi_i(0)}, \phi_{i+1}(0), \dots, \phi_n(0)) \right] (\dot{\phi}_i(0)).$$

By direct computation one shows that the following diagram of cochain maps commutes:

$$\begin{array}{ccccc} C^m(\mathfrak{g}; \mathbb{R}) & \xrightarrow{\text{var}} & C^{n-1}(\mathfrak{g}; \mathfrak{g}^\vee) & \xrightarrow{F} & C^n(\widetilde{\mathfrak{g}}; \mathbb{R}) \\ & \searrow & & & \downarrow \text{Char}_{\nabla_t} \\ & & & & \Omega_{dR}^n(E) \\ & & \text{Var}_{\nabla_t} & \nearrow & \end{array}$$

Let us now relativize the construction above. We have fixed a relative cocycle  $(\omega, \{\beta\}) \in C^n(\mathfrak{g}, \{\mathfrak{b}\})$ , a 1-parameter family of connections  $\nabla_t$  on a manifold  $E$ , and a family of closed submanifolds  $\{A\}$  in  $E$ . Assume that for each value of  $t$  the restriction of  $\nabla_t$  to  $A$  takes values in the Lie subalgebra  $\mathfrak{b}_t^A \in \{\mathfrak{b}\}$ . Then for each value of the parameter  $t$  we have as data a relative de Rham cocycle with absolute part:

$$(X_1, \dots, X_n) \mapsto \omega(\nabla_t X_1, \dots, \nabla_t X_n),$$

and relative part given on each submanifold  $A$  by:

$$(Y_1, \dots, Y_{n-1}) \mapsto \beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}).$$

The instant variation of this class is given by computing the usual limit. For the absolute part  $\omega$  we get the same result as in the non-relative case:

$$\text{Var}_{\nabla_t}(\omega)$$

For the relative part we have to compute the limit as  $t \rightarrow 0$  of:

$$(2) \quad \Delta(\beta, t) = \frac{1}{t} (\beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}) - \beta_0^A(\nabla_0 Y_1, \dots, \nabla_0 Y_{n-1})) \quad (*)$$

Here we are stuck as the usual tricks that lead to a Leibniz type derivation formula in this case do not work: the problem lies in the fact that the class  $\beta_t^A$  also depends on the time  $t$ . To overcome this difficulty we will impose the following coherence condition on the connection with respect to the family of Lie subalgebras  $\mathfrak{b}_t^A$ :

**Definition 2.** Assume  $\mathfrak{g}$  is the Lie algebra of a connected Lie group  $G$ . Consider on a manifold  $E$  with a family of submanifolds  $A$  a one parameter family of connections  $\nabla_t$ . Assume that for each  $A$ , the restriction  $\nabla_0|_A$  lies into the Lie subalgebra  $\mathfrak{b}^A$ . We say that the connection varies coherently along the submanifolds  $A$  with respect to the family  $\{\mathfrak{b}\}$  if and only if the following holds:

*There is a subgroup  $H \subset G$  such that for each subspace  $A$  there exists a smooth 1-parameter family of elements of  $H$ ,  $h_t$  with  $h_0 = \text{Id}$ , such that for each value of the parameter  $t$  the connection  $\widetilde{\nabla}_t^A = \text{Ad}_{h_t} \nabla_t|_A$  takes values in the Lie subalgebra at the origin  $\mathfrak{b}^A$ .*

This condition will force us to restrict our treatment of the variation of a relative characteristic class in two ways:



- (1) Firstly we will only consider classes whose global part is an  $H$ -invariant cocycle, where  $H$  is the group defined above.
- (2) Secondly, given a connection that varies coherently along the submanifolds  $A$  with respect to the family  $\{\mathfrak{b}\}$ , we will ask for the relative part of the cocycle to satisfy the coherence condition:

$$\forall(Y_1, \dots, Y_{n-1}) \quad \beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}) = \beta_0^A(\tilde{\nabla}_t^A Y_1, \dots, \tilde{\nabla}_t^A Y_{n-1}).$$

**Definition 3.** We say that the characteristic class varies coherently with the connection if the previous two conditions are satisfied.

Under this assumption we can pursue the computation in (2) above:

$$\begin{aligned} \lim_{t \rightarrow 0} \Delta(\beta, t) &= \lim_{t \rightarrow 0} \frac{1}{t} (\beta_t^A(\nabla_t Y_1, \dots, \nabla_t Y_{n-1}) - \beta_0^A(\nabla_0 Y_1, \dots, \nabla_0 Y_{n-1})) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (\beta_0^A(\tilde{\nabla}_t^A Y_1, \dots, \tilde{\nabla}_t^A Y_{n-1}) - \beta_0^A(\tilde{\nabla}_0^A Y_1, \dots, \tilde{\nabla}_0^A Y_{n-1})) \\ &= \sum_{j=1}^{n-1} \beta_0^A(\tilde{\nabla}_0^A Y_1, \dots, \tilde{\nabla}_0^A Y_{j-1}, \dot{\tilde{\nabla}}_0^A Y_j, \tilde{\nabla}_0^A Y_{j+1}, \dots, \tilde{\nabla}_t^A Y_{n-1}) \\ &= \text{Var}_{\tilde{\nabla}_t}(\beta_0^A) \end{aligned}$$

Observe that, since  $\omega$  is  $H$ -invariant, for any vector fields  $(X_1, \dots, X_n)$  on  $E$  we have

$$\omega(\nabla_t X_1, \dots, \nabla_t X_n) = \omega(\tilde{\nabla}_t X_1, \dots, \tilde{\nabla}_t X_n),$$

and in particular as differential forms

$$d \text{Var}_{\tilde{\nabla}_t}(\beta_t^A) = j_A^*(\text{Var}_{\nabla_t}(\omega)) = j_A^*(\text{Var}_{\tilde{\nabla}_t}(\omega)),$$

where  $j_A : A \hookrightarrow E$  is the inclusion. Hence the data

$$(\text{Var}_{\nabla_t}(\omega), \{\text{Var}_{\tilde{\nabla}_t}(\beta^A)\}) = \text{Var}_{\nabla_t, \{\tilde{\nabla}_t\}}(\omega, \{\beta\})$$

is indeed a relative differential form on  $(E, \{A\})$ .

We will now relativize the maps involved in Fuks factorization.

**Lemma 1.** Let  $G$  be a connected Lie group,  $\mathfrak{g}$  its Lie algebra and  $H \subset G$  a subgroup. Then the cochain complexes  $C^*(\mathfrak{g}; \mathbb{R})$ ,  $C^*(\mathfrak{g}; \mathfrak{g}^\vee)$  and  $C^*(\tilde{\mathfrak{g}}; \mathbb{R})$  are cochain complexes of  $H$ -modules, where the action of  $H$  is induced by its adjoint action on  $\mathfrak{g}$ . Moreover the maps  $\text{var}$  and  $F$  are compatible with the action of  $H$ .

*Proof.* This is an immediate consequence of the fact that the above chain complexes are functorial in the variable  $\mathfrak{g}$  and the adjoint action is by automorphisms of Lie algebras.  $\square$

**Notation 2.** Denote by  $C_H^*(\mathfrak{g}; \mathbb{R})$ ,  $C_H^*(\mathfrak{g}; \mathfrak{g}^\vee)$  and  $C_H^*(\tilde{\mathfrak{g}})$  the subspace of fixed points under the action of  $H$  of the vector spaces  $C^*(\mathfrak{g})$ ,  $C^*(\mathfrak{g}; \mathfrak{g}^\vee)$  and  $C^*(\tilde{\mathfrak{g}}; \mathbb{R})$  respectively.

**Definition 4.** Denote by

$$C_H^*(\mathfrak{g}; \{\mathfrak{b}\}) = \text{Cone}(C_H^*(\mathfrak{g}; \mathbb{R}) \rightarrow \prod C^*(\mathfrak{b}; \mathbb{R}))$$

$$C_H^*(\mathfrak{g}, \{\mathfrak{b}\}; \mathfrak{g}^\vee, \{\mathfrak{b}^\vee\}) = \text{Cone}(C_H^*(\mathfrak{g}; \mathfrak{g}^\vee) \rightarrow \prod C^*(\mathfrak{b}; \mathfrak{b}^\vee))$$

the cones taken along the maps induced by the inclusions  $\mathfrak{b} \rightarrow \mathfrak{g}$ .

Notice that in the above definition we do not ask a priori the chains on the Lie algebras  $\mathfrak{b}$  to be invariant of any sort.

**Proposition 2.** *Via the cone construction the chain maps  $\text{var}$  and  $F$  induce relative chain maps:*

$$\text{var}: C_H^*(\mathfrak{g}, \{\mathfrak{b}\}) \longrightarrow C_H^{*-1}(\mathfrak{g}, \{\mathfrak{b}\}; \mathfrak{g}^\vee, \{\mathfrak{b}^\vee\})$$

and

$$F: C_H^{*-1}(\mathfrak{g}, \{\mathfrak{b}\}; \mathfrak{g}^\vee, \{\mathfrak{b}^\vee\}) \longrightarrow C_H^*(\widetilde{\mathfrak{g}}, \{\widetilde{\mathfrak{b}}\})$$

*Proof.* This follows from the functoriality of the cone construction and the commutativity, for any Lie subalgebra  $\mathfrak{b} \subset \mathfrak{g}$ , of the following two squares:

$$\begin{array}{ccc} C_H^*(\mathfrak{g}; \mathbb{R}) & \longrightarrow & C_H^{*-1}(\mathfrak{g}; \mathfrak{g}^\vee) \\ \downarrow & & \downarrow \\ C^*(\mathfrak{b}; \mathbb{R}) & \longrightarrow & C^{*-1}(\mathfrak{b}; \mathfrak{b}^\vee) \end{array}$$

and

$$\begin{array}{ccc} C_H^{*-1}(\mathfrak{g}; \mathfrak{g}^\vee) & \longrightarrow & C_H^*(\widetilde{\mathfrak{g}}; \mathbb{R}) \\ \downarrow & & \downarrow \\ C^{*-1}(\mathfrak{b}; \mathfrak{b}^\vee) & \longrightarrow & C^*(\widetilde{\mathfrak{b}}; \mathbb{R}) \end{array}$$

□

**Proposition 3.** *With the notations of Definition 2, the map  $\text{Char}_{\nabla_t}: C_H^*(\widetilde{\mathfrak{g}}; \mathbb{R}) \rightarrow \Omega_{dR}^*(E)$  and  $\text{Char}_{\widetilde{\nabla}_t^A}: C_H^*(\widetilde{\mathfrak{b}^A}; \mathbb{R}) \rightarrow \Omega_{dR}^*(A)$  induce a map in relative cohomology*

$$\text{Char}_{\nabla_t, \{\widetilde{\nabla}_t^A\}}: C_H^*(\widetilde{\mathfrak{g}}, \{\widetilde{\mathfrak{b}^A}\}) \rightarrow \Omega_{dR}^*(E, \{A\})$$

which is compatible with the restrictions and inflation maps, where

$$C_H^*(\widetilde{\mathfrak{g}}, \{\widetilde{\mathfrak{b}^A}\}) = \text{Cone} \left( C_H^*(\widetilde{\mathfrak{g}}; \mathbb{R}) \hookrightarrow C^*(\widetilde{\mathfrak{g}}; \mathbb{R}) \xrightarrow{\text{rest}_t} \prod_A C^*(\widetilde{\mathfrak{b}^A}; \mathbb{R}) \right)$$

*Proof.* By functoriality of the cone construction, it is enough to show that for each  $A$  the following diagram where the vertical maps are the restriction maps commutes:

$$\begin{array}{ccc} C_H^*(\widetilde{\mathfrak{g}}; \mathbb{R}) & \xrightarrow{\text{Char}_{\nabla_t}} & \Omega_{dR}^*(E) \\ \downarrow & & \downarrow \\ C^*(\widetilde{\mathfrak{b}^A}; \mathbb{R}) & \xrightarrow{\text{Char}_{\widetilde{\nabla}_t^A}} & \Omega_{dR}^*(A) \end{array}$$

which is achieved by a trivial diagram chasing. □

Summing up the results in this section the discussion on the relative variation maps we have shown that:

**Theorem 3.** *Let  $(\omega, \{\beta\}) \in C_H^*(\mathfrak{g}, \{\mathfrak{b}\})$  vary coherently along a connection  $\nabla_t$  on  $E$ .*

*The variation chain map  $\text{Var}: C^*(\mathfrak{g}; \mathbb{R}) \rightarrow \Omega_{dR}^*(E)$  induces via the cone construction a relative variation chain map*

$$\text{Var}_{\nabla_t, \{\widetilde{\nabla}_t\}}: C_H^*(\mathfrak{g}, \{\mathfrak{b}\}) \rightarrow \Omega_{dR}^*(E, \{A\})$$

*whose induced map in cohomology computes the derivative at  $t = 0$  of the cohomology classes  $\text{Char}_{\nabla_t}(\omega, \{\beta\}) \in H_{dR}^*(E, \{A\})$ .*

We also have a Fuks type factorization of the variation map:

**Theorem 4.** *The relative Variation map factors as:*

$$\begin{array}{ccc}
 C_H^*(\mathfrak{g}, \{\mathfrak{b}\}) & \xrightarrow{\text{var}} C_H^{*-1}(\mathfrak{g}, \{\mathfrak{b}\}; \mathfrak{g}^\vee, \{\mathfrak{b}^\vee\}) & \xrightarrow{F} C_H^*(\tilde{\mathfrak{g}}, \{\tilde{\mathfrak{b}}\}) \\
 & \searrow \text{Var} & \downarrow \text{Char}_{\nabla_t, \{\bar{\nabla}_t\}} \\
 & & \Omega_{dR}^*(E, \{A\})
 \end{array}$$

and this factorization is compatible with the restriction and connecting homomorphisms.

#### 4. THE VOLUME OF A REPRESENTATION

**4.1. Set-up and notations.** Let us start with some definitions and notations involving the structure of the groups  $\text{SL}_n(\mathbb{C})$ . We will see these groups as real Lie groups. Recall then that for each  $n \geq 2$ , the group  $\text{SU}(n) \subset \text{SL}_n(\mathbb{C})$  is a maximal compact subgroup. Let  $D_n \subset \text{SL}_n(\mathbb{C})$  denote the subgroup of diagonal matrices, then  $D_n \cap \text{SU}(n) = T$  is a maximal real torus isomorphic to  $(\mathbb{S}^1)^{n-1}$ . By definition a Borel subgroup of  $\text{SL}_n(\mathbb{C})$  is a maximal solvable subgroup; the Borel subgroups are also the stabilizers of complete flags in  $\mathbb{C}^n$ , the Gram-Schmidt process then shows that the subgroup  $\text{SU}(n)$  acts transitively on complete flags, and hence that all Borel subgroups are pairwise conjugated in  $\text{SL}_n(\mathbb{C})$  by elements in  $\text{SU}(n)$ .

We fix as our model Borel subgroup  $B \subset \text{SL}_n(\mathbb{C})$  the subgroup of upper-triangular matrices, in particular the transitive action by conjugation of  $\text{SU}(n)$  on the set of all Borel subgroups provides each of these with a specified choice of a maximal compact subgroup. Denote by  $U_n \subset B$  the subgroup of unipotent matrices; this is a normal subgroup and gives  $B$  the structure of a semi-direct product  $B = U_n \rtimes D_n$ .

Again by the Gram-Schmidt process, the inclusion  $B \hookrightarrow \text{SL}_n(\mathbb{C})$  induces an homeomorphism of homogeneous manifolds  $B/T \simeq \text{SL}_n(\mathbb{C})/\text{SU}(n)$ . For  $n = 2$ , this symmetric space is hyperbolic space. For normalization purposes, let us recall that

$$(3) \quad \begin{pmatrix} e^{(l+i\theta)/2} & 0 \\ 0 & e^{-(l+i\theta)/2} \end{pmatrix} = \exp \begin{pmatrix} (l+i\theta)/2 & 0 \\ 0 & -(l+i\theta)/2 \end{pmatrix}$$

acts on  $\text{SL}_2(\mathbb{C})/\text{SU}(2) \simeq \mathcal{H}^3$  as the composition of a loxodromic isometry of translation length  $l$  composed with a rotation of angle  $\theta$  along the same axis, cf. [16, §12.1].

Let  $\pi$  denote the fundamental group of  $M$ , the compact three-manifold with nonempty boundary, whose interior is hyperbolic of finite volume. The  $k \geq 1$  boundary components are tori,  $\partial M = T_1^2 \sqcup T_2^2 \cdots \sqcup T_k^2$ . For each boundary component of  $M$  fix a path from the basepoint of  $M$  to the boundary, this gives us a definite choice of a peripheral system  $P_1, \dots, P_k$  in  $\pi$ , where  $P_i \simeq \pi_1(T_i^2)$ .

Let's now fix a representation  $\rho : \pi \rightarrow \text{SL}_n(\mathbb{C})$  for some  $n \geq 2$ . Since each peripheral subgroup  $P_i$  is abelian and the Borel subgroups of  $\text{SL}_n(\mathbb{C})$  are maximal solvable subgroups, the image of the restriction of  $\rho$  to  $P_i$  lies in a Borel subgroup. Fix for each peripheral subgroup  $P_i$  such a Borel subgroup  $B_i$ .

**4.2. Some known results in bounded and continuous cohomology.** The continuous cohomology of the groups  $\text{SL}_n(\mathbb{C})$  has a rather simple structure:

**Proposition 4.** [3] *Let  $n \geq 1$  be an integer, then  $H_c^*(\text{SL}_n(\mathbb{C}))$  is an exterior algebra:*

$$H_c^*(\text{SL}_n(\mathbb{C})) = \bigwedge \langle x_{j,n} \mid 1 \leq j \leq n \rangle$$

over so-called Borel classes  $x_{j,n}$  of degree  $2j + 1$ . These classes are stable, if  $j_n : \text{SL}_n(\mathbb{C}) \rightarrow \text{SL}_{n+1}(\mathbb{C})$  denotes the inclusion in the upper left corner, then for  $j \leq n$ ,  $j_n^*(x_{n+1,j}) = x_{n,j}$ .

**Remark 1.** For  $\text{SL}_2(\mathbb{C})$  the Borel class  $x_1$  is also known as the hyperbolic volume class and we denote it by  $\text{vol}_{\mathcal{H}^3}$ . It is completely determined by stability and the requirement that on  $\text{SL}_2(\mathbb{C})$  it

is represented by the cocycle

$$(A, B, C, D) \mapsto \int_{(A*, B*, C*, D*)} d \operatorname{vol}_{\mathcal{H}^3}$$

where  $(A*, B*, C*, D*)$  denotes the hyperbolic oriented tetrahedron with geodesic faces spanned by the four images of the base point  $* \in \mathcal{H}^3$  by  $A, B, C$  and  $D$  respectively and  $d \operatorname{vol}_{\mathcal{H}^3}$  is the hyperbolic volume form. Notice that this cocycle is bounded by the maximal volume of an ideal tetrahedron.

Compared to the relatively simple structure of the continuous cohomology, the continuous-bounded cohomology of  $\operatorname{SL}_n(\mathbb{C})$  is considerably more complicated and largely unknown, see Monod [18]. Nevertheless, fitting well our purposes we have the following:

**Proposition 5.** [19] *The canonical comparison map  $H_{cb}^3(\operatorname{SL}_n(\mathbb{C})) \rightarrow H_c^3(\operatorname{SL}_n(\mathbb{C}))$  is surjective.*

For continuous bounded we have also the following crucial feature, which applies in particular to the Borel and unipotent subgroups of  $\operatorname{SL}_n(\mathbb{C})$ :

**Proposition 6.** *Let  $G$  denote an amenable Lie group, e.g. abelian or solvable, then  $H_{cb}^*(G) = 0$  for  $* > 0$ .*

We are now ready to define the volume of our representation  $\rho : \pi \rightarrow \operatorname{SL}_n(\mathbb{C})$ . The long exact sequence in continuous cohomology for the pair  $(\operatorname{SL}_n(\mathbb{C}), \{B_i\})$ , where  $\{B_i\}$  stands for the family of Borel subgroups we have fixed together with Proposition 6 gives immediately:

**Proposition 7.** *For  $* \geq 2$ , the map induced by forgetting the relative part induces an isomorphism:*

$$H_{cb}^*(\operatorname{SL}_n(\mathbb{C}), \{B_i\}) \xrightarrow{\sim} H_{cb}^*(\operatorname{SL}_n(\mathbb{C})).$$

Comparing continuous cohomology and bounded continuous cohomology for the pair  $(\operatorname{SL}_n(\mathbb{C}), \{B_i\})$  gives us a commutative diagram:

$$\begin{array}{ccc} H_{cb}^3(\operatorname{SL}_n(\mathbb{C}), \{B_i\}) & \xrightarrow{\sim} & H_{cb}^3(\operatorname{SL}_n(\mathbb{C})) \\ \downarrow & & \downarrow \\ H_c^3(\operatorname{SL}_n(\mathbb{C}), \{B_i\}) & \longrightarrow & H_c^3(\operatorname{SL}_n(\mathbb{C})) \end{array}$$

This shows that the continuous-bounded cohomology class  $\operatorname{vol}_{\mathcal{H}}$  has a canonical representative as a continuous bounded relative class  $\operatorname{vol}_{\mathcal{H}, \partial} \in H_c^3(\operatorname{SL}_n(\mathbb{C}), \{B_i\})$ .

By construction the representation  $\rho$  induces a map of pairs  $\rho : (\pi, \{P_i\}) \rightarrow (\operatorname{SL}_n(\mathbb{C}), \{B_i\})$ , hence by functoriality we have an induced map in continuous cohomology

$$H_c^3(\operatorname{SL}_n(\mathbb{C}), \{B_i\}) \xrightarrow{\rho^*} H_c^3(\pi, \{P_i\}).$$

But for discrete groups continuous cohomology and ordinary group cohomology coincide, so we have a well-defined class, up to a possible ambiguity given by the choice of the Borel subgroups  $B_i$ ,

$$\rho^*(\operatorname{vol}_{\mathcal{H}, \partial}) \in H^3(\pi, \{P_i\}).$$

**Proposition 8.** *The class  $\rho^*(\operatorname{vol}_{\mathcal{H}, \partial}) \in H_c^3(\pi, \{P_i\})$  is independent of the possible choice of a different family of Borel subgroups  $\{B_i\}$ .*

*Proof.* Let us assume for clarity that we have two possible choices  $B_j$  and  $B'_j$  for the Borel subgroup that contains  $\rho(P_j)$ , and that we make a unique choice for the rest of the peripheral subgroups. We denote the two families of subgroups by  $\{B_{i \neq j}, B_j\}$  and  $\{B_{i \neq j}, B'_j\}$ . Because Borel subgroups are closed in  $\operatorname{SL}_n(\mathbb{C})$ , their intersection, as  $B_i \cap B'_i$ , is also amenable. The restriction of  $\rho$  to the

peripheral subgroup  $P_j$  factors in both cases through this intersection, so we have a commutative diagram of group homomorphisms:

$$\begin{array}{ccccc}
 & & & & (\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B_j\}) \\
 & & \nearrow & & \\
 (\pi, \{P_{i \neq j}, P_j\}) & \longrightarrow & (\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B_j \cap B'_j\}) & \nearrow & \\
 & \searrow & & \searrow & \\
 & & & & (\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B'_j\})
 \end{array}$$

Together with the forgetful isomorphisms to the absolute cohomology of  $\mathrm{SL}_n(\mathbb{C})$  given by the fact the subgroups involved are all amenable, we have a commutative diagram:

$$\begin{array}{ccccc}
 & & & & H_{cb}^3(\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B_j\}) \\
 & & \nearrow & & \parallel \\
 H_{cb}^3(\pi, \{P_{i \neq j}, P_i\}) & \longleftarrow & H_{cb}^3(\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B_j \cap B'_j\}) & \xlongequal{\quad} & H_{cb}^3(\mathrm{SL}_n(\mathbb{C})) \\
 & \nwarrow & & \nwarrow & \parallel \\
 & & & & H_{cb}^3(\mathrm{SL}_n(\mathbb{C}), \{B_{i \neq j}, B'_j\})
 \end{array}$$

and this finishes the proof.  $\square$

Now  $M$  is a  $K(\pi, 1)$  and each boundary component is a  $K(P_i, 1)$  for the corresponding peripheral subgroup, in particular  $H^3(\pi, \{P_i\}) \simeq H^3(M; \partial M) \simeq \mathbb{R}$  by Theorem 2, due to Bieri and Eckmann, and this leads to our compact definition of the volume of a representation (for a more precise statement see Definition 6):

**Definition 5.** *Let  $\rho: \pi \rightarrow \mathrm{SL}_n(\mathbb{C})$  be a representation of the fundamental group of a finite volume hyperbolic 3-manifold. Then*

$$\mathrm{Vol}(\rho) = \langle \rho^*(\mathrm{vol}_{\mathcal{H}, \partial}), [M, \partial M] \rangle.$$

In [7] Bucher, Burger and Iozzi prove that the volume of a representation  $\pi \rightarrow \mathrm{SL}_n(\mathbb{C})$  is maximal at the composition of the irreducible representation  $\mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_n(\mathbb{C})$  with a lift of the holonomy. Their definition, as ours, relies on continuous bounded cohomology and are clearly equivalent: their transfer argument is replaced here by an isomorphism through a relative cohomology group. The passage through continuous cohomology seems for the moment rather useless, it will however be crucial in our next set: the study of the variation of the volume.

## 5. VARIATION OF THE VOLUME CLASS

We will follow Reznikov's idea [20] to prove rigidity of the volume in the compact case. We will first show that the volume class can be viewed as a characteristic class on the total space of the flat bundle defined by the representation, then find explicit relative cocycles representing  $\mathrm{vol}_{\mathcal{H}, \partial}$  and finally apply the machinery of Section 3.

Let us start by some more notations. In the previous section we defined a series of Lie subgroups of  $\mathrm{SL}_n(\mathbb{C})$ , we now pass to their Lie algebras, all viewed as real Lie algebras.

Lie group	Lie algebra	Description as subgroup
$\mathrm{SL}_n(\mathbb{C})$	$\mathfrak{sl}_n$	
$\mathrm{SU}(n)$	$\mathfrak{su}_n$	Fixed maximal compact subgroup
$B$	$\mathfrak{b}_n$	Fixed Borel subgroup of upper triangular matrices
$D_n$	$\mathfrak{h}_n + i\mathfrak{h}_n$	Subgroup of diagonal matrices in $B$
$T = D_n \cap \mathrm{SU}(n)$	$\mathfrak{h}_n$	Maximal torus in $\mathrm{SU}(n)$ (and in $B$ and $\mathrm{SL}_n(\mathbb{C})$ )
$U_n$	$\mathfrak{ut}_n$	Subgroup of unipotent elements in $B$ .

For explicit formulas, we will need a concrete basis for the real Lie algebra  $\mathfrak{su}_n$ . Recall that  $\mathfrak{su}_n = \{X \in M_n(\mathbb{C}) \mid X + {}^t\overline{X} = 0 \text{ and } \mathrm{tr}(X) = 0\}$ .

There is a standard  $\mathbb{R}$ -basis of  $\mathfrak{su}_2$ , orthogonal with respect to the Killing form:

$$h = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}.$$

From this we can construct an analogous basis for  $\mathfrak{su}_n$ ; we only give here the non-zero entries of the matrices.

- (1) For an integer  $1 \leq s \leq n-1$  let  $h_s$  denote the matrix with a coefficient  $i/2$  in diagonal position  $s$  and a coefficient  $-i/2$  in diagonal position  $n$ . It will be convenient to denote  $h_{st} = h_s - h_t$ .
- (2) For any pair of integers  $1 \leq s < t \leq n$  let  $e_{st}$  has coefficient row  $s$  and column  $t$  equal to  $1/2$  and coefficient row  $t$  and column  $s$  equal to  $-1/2$ .
- (3) For any pair of integers  $1 \leq s < t \leq n$  let  $f_{st}$  denote the matrix which has coefficient row  $s$  and column  $t$  equal to  $i/2$  and coefficient row  $t$  and column  $s$  equal to  $i/2$ .

Notice that the matrices  $h_s$  generate the Lie subalgebra  $\mathfrak{h}$ , the Lie algebra of the real torus  $T$ . The dual basis will be denoted by  $h_s^\vee, e_{st}^\vee, h_{st}^\vee$ . With these conventions, for  $n = 2$ ,  $h = h_1, e = e_{12}$  and  $f = f_{12}$ .

Analogously, for  $\mathfrak{b}$ , the Lie algebra of upper triangular matrices with zero trace, we have a basis made of the matrices  $h_s, ih_s, 1 \leq s \leq n-1$ , and for  $1 \leq k < l \leq n$ , the matrices  $ur_{kl}$  (upper real) which are equal to 1 in row  $k$  and column  $l$  and  $ui_{kl} = iur_{kl}$  (upper imaginary matrices). We have  $ur_{kl} = e_{kl} - i f_{kl}$  and  $ui_{kl} = i e_{kl} + f_{kl}$ .

The following result provides us with the right cochain complex in which to find our cocycle representatives; beware that the relative cohomology of Lie algebras in the statement is not the one we defined in Section 2, but the classical one as defined for instance in Weibel [21, Chap. 7].

**Proposition 9** ([14] Van Est isomorphism for trivial coefficients). *Let  $G$  be a connected real Lie group. Denote by  $\mathfrak{g}$  its Lie algebra and  $\mathfrak{k}$  the Lie algebra of a maximal compact subgroup  $K \subset G$ . Then for all  $m$  there is a canonical isomorphism  $H_c^m(G; \mathbb{R}) \simeq H^m(\mathfrak{g}, \mathfrak{k}; \mathbb{R})$ . More precisely the de Rham cochain complex of left-invariant differential forms*

$$0 \longrightarrow \Omega_{dR}^0(G/K)^G \longrightarrow \cdots \longrightarrow \Omega_{dR}^n(G/K)^G \longrightarrow \cdots$$

*computes both cohomologies.*

Recall that the volume class comes from a bounded cohomology class, so their de Rham representatives will be special and can be explicitly detected thanks to the following result of Burger and Iozzi (Prop. 3.1 in [8])

**Proposition 10.** [8] *Let  $G$  be a connected semi-simple Lie group with finite center, let  $K$  be a maximal compact subgroup, let  $G/K$  the associated symmetric space and let  $L \subset G$  be any closed subgroup. Then there exists a map*

$$\delta_{\infty, L}^* : H_{cb}^*(L; \mathbb{R}) \rightarrow H^*(\Omega_{dR, \infty}(G/K)^L)$$

such that the diagram:

$$\begin{array}{ccccc}
 H_{cb}^*(L; \mathbb{R}) & \xrightarrow{c_L^*} & H_c^*(L; \mathbb{R}) & \xleftarrow{\simeq} & H^*(\Omega(G/K)^L) \\
 & \searrow \delta_{\infty, L} & & \nearrow i_{\infty, L} & \\
 & & H^*(\Omega_{dR, \infty}(G/K)^L) & & 
 \end{array}$$

commutes, where  $\Omega_{dR, \infty}(G/K)$  is the de Rham complex of bounded differential forms with bounded differential and  $i_{\infty, L}$  is the map induced in cohomology by the inclusion of complexes  $\Omega_{dR, \infty}(G/K) \hookrightarrow \Omega_{dR}(G/K)$ .

**5.1. A relative cocycle representing  $\text{vol}_{\mathcal{H}, \partial}$ .** By the Van Est isomorphism, we look for a cocycle representing the absolute part in  $\Omega_{dR}^3(\text{SL}_n(\mathbb{C})/\text{SU}(n))^{SL_n(\mathbb{C})}$  and for the relative part in  $\Omega_{dR}^2(B/T)^B$ .

**5.1.1. The absolute part.** Denote by  $K_{\mathfrak{sl}_n}^{\mathbb{R}}$  the real Killing form of the real Lie algebra  $\mathfrak{sl}_n$ . With respect to this form we have an orthogonal decomposition  $\mathfrak{sl}_n = \mathfrak{su}_n \oplus i\mathfrak{su}_n$ . We denote by:

$$\begin{array}{ccc}
 pr_{\mathfrak{su}_n} : \mathfrak{sl}_n & \longrightarrow & \mathfrak{su}_n \\
 A & \longmapsto & \frac{1}{2}(A - t\overline{A})
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 pr_{i\mathfrak{su}_n} : \mathfrak{sl}_n & \longrightarrow & i\mathfrak{su}_n \\
 A & \longmapsto & \frac{1}{2}(A + t\overline{A})
 \end{array}$$

the canonical projections.

The behavior of these projections with respect to the Lie bracket is given by:

$$(4) \quad pr_{\mathfrak{su}}([a, b]) = [pr_{\mathfrak{su}}a, pr_{\mathfrak{su}}b] + [pr_{i\mathfrak{su}}a, pr_{i\mathfrak{su}}b]$$

$$(5) \quad pr_{i\mathfrak{su}}([a, b]) = [pr_{\mathfrak{su}}a, pr_{i\mathfrak{su}}b] + [pr_{i\mathfrak{su}}a, pr_{\mathfrak{su}}b]$$

The tangent space at the class of Id of the symmetric space  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$  is canonically identified with  $i\mathfrak{su}_n$ , and the induced action of  $\text{SU}(n)$  on this tangent space is easily checked to be the adjoint action. Let us now consider the following rescaling of the *complex* Killing form on  $\mathfrak{sl}_n$ ,  $A, B \rightsquigarrow \text{tr}(AB)$ . This gives rise to a complex valued alternating 3 form, sometimes known as the (here rescaled) Cartan-Killing form:  $CK_{\mathfrak{sl}_n}^{\mathbb{C}} : (A, B, C) \mapsto \text{tr}(A[B, C])$ . It is folklore knowledge that “the hyperbolic volume is the imaginary part of this Cartan-Killing form” (see Yoshida [22] for a precise statement when  $n = 2$  or Reznikov [20]); let us turn this into a precise statement. We fix our attention in the following part of the de Rham complex:

$$\Omega_{dR}^2(\text{SL}_n(\mathbb{C})/\text{SU}(n))^{SL_n(\mathbb{C})} \rightarrow \Omega_{dR}^3(\text{SL}_n(\mathbb{C})/\text{SU}(n))^{SL_n(\mathbb{C})} \rightarrow \Omega_{dR}^4(\text{SL}_n(\mathbb{C})/\text{SU}(n))^{SL_n(\mathbb{C})}$$

**Lemma 2.** *The vector space  $\Omega_{dR}^2(\text{SL}_n(\mathbb{C})/\text{SU}(n))^{SL_n(\mathbb{C})}$  is trivial.*

*Proof.* By transitivity of the action, an alternating 2-form on the homogeneous space  $\text{SL}_n(\mathbb{C})/\text{SU}(n)$  is completely determined by what happens at the class of the identity, i.e. by a unique element in  $(\bigwedge^2(i\mathfrak{su}_n)^{\vee})^{\text{SU}(n)}$ . As  $\text{SU}(n)$ -modules  $i\mathfrak{su}^{\vee}$  and  $\mathfrak{su}^{\vee}$  are isomorphic, and via the real Killing form on  $\text{SU}(n)$ , a symmetric non-degenerate form, the Lie algebra  $\mathfrak{su}$  and its dual are also isomorphic  $\text{SU}(n)$ -modules. So to prove the statement it is enough to show that  $(\bigwedge^2 \mathfrak{su}_n)^{\text{SU}(n)} = 0$ . Let  $\phi : \mathfrak{su}(n) \wedge \mathfrak{su}(n) \rightarrow \mathbb{R}$  be a skew-symmetric invariant form. Invariance by the adjoint action of  $\text{SU}(n)$  is equivalent to:

$$\phi([X, Y], Z) + \phi(Y, [X, Z]) = 0 \quad \forall X, Y, Z \in \mathfrak{su}(n).$$

Combined with skew-symmetry of both  $\phi$  and the Lie bracket, this equality yields

$$\phi([X, Y], Z) = \phi([X, Z], Y) = -\phi([Z, X], Y) \quad \forall X, Y, Z \in \mathfrak{su}(n).$$

Namely,  $\phi([X, Y], Z)$  changes the sign when the entries  $X, Y, Z \in \mathfrak{su}(n)$  are cyclically permuted, therefore it vanishes. Then  $\phi = 0$  because  $\mathfrak{su}(n)$  is simple.  $\square$

For a manifold  $X$ , denote by  $Z_{dR}^n(X) \subset \Omega_{dR}^n(X)$  the subspace of closed forms.

**Corollary 1.** *The canonical quotient map  $Z_{dR}^3(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))^{\mathrm{SL}_n(\mathbb{C})} \rightarrow H_c^3(\mathrm{SL}_n(\mathbb{C})) \simeq \mathbb{R}$  is an isomorphism.*

Since  $H_c^3(\mathrm{SL}_n(\mathbb{C})) \simeq \mathbb{R}$  by Borel's computations, there is a unique closed form on  $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$  that represents the class  $\mathrm{vol}_{\mathcal{H}}$ . There is an obvious candidate for such a form, it is given on the tangent space at  $\mathrm{Id}$  by:

$$\begin{aligned} \bigwedge^3 \mathfrak{isu}_n &\longrightarrow \mathbb{R} \\ (A, B, C) &\longmapsto 2i \operatorname{tr}(A[B, C]) = -2\Im \operatorname{tr}(A[B, C]) \end{aligned}$$

Then  $\varpi_n: \mathfrak{sl}_n \rightarrow \mathbb{R}$  is the composition of the projection  $pr_{\mathfrak{isu}}: \mathfrak{sl}_n \rightarrow \mathfrak{isu}_n$  with this form:

$$\begin{aligned} \varpi: \bigwedge^3 \mathfrak{sl}_n &\longrightarrow \mathbb{R} \\ (A, B, C) &\longmapsto 2i \operatorname{tr}(pr_{\mathfrak{isu}}(A)[pr_{\mathfrak{isu}}(B), pr_{\mathfrak{isu}}(C)]) \end{aligned}$$

That this form is alternating and invariant under the adjoint action of  $\mathrm{SU}(n)$  is an immediate consequence of the fact that the Cartan-Killing form  $(A, B, C) \mapsto \operatorname{tr}(A[B, C]) = \operatorname{tr}(ABC - ACB)$  is alternating and  $\mathrm{SU}(n)$ -invariant, and that the adjoint action of  $\mathrm{SU}(n)$  respects the decomposition of  $\mathfrak{sl}_n = \mathfrak{su}_n \oplus \mathfrak{isu}_n$ . Observe that by construction this form is compatible with the inclusions  $\mathfrak{sl}_n \rightarrow \mathfrak{sl}_{n+1}$ : if we denote the form defined by  $\mathfrak{sl}_n$  by  $\varpi_n$  then  $\varpi_{n+1}|_{\mathfrak{sl}_n} = \varpi_n$ , in line of the stability result of Borel in degree 3. We only have to check that this is a cocycle when viewed as a classical relative cocycle in Lie algebra cohomology of  $\mathfrak{sl}_n/\mathfrak{su}_n = \mathfrak{isu}_n$  (i.e. gives rise to a closed form), that it is not trivial and fix the normalization constant; this will be done by comparing it with the hyperbolic volume form for  $n = 2$ .

**Lemma 3.** *The alternating 3-form  $\varpi \in \operatorname{Hom}(\bigwedge^3 \mathfrak{sl}_n, \mathbb{R})$  is a cocycle.*

*Proof.* By definition of the differential in the Cartan-Chevalley complex see Weibel [21, Chap. 7], and since  $[\mathfrak{isu}_n, \mathfrak{isu}_n] \subset \mathfrak{su}_n$ , the differential in this cochain complex is in fact trivial, so any element in  $\operatorname{Hom}(\bigwedge^3 \mathfrak{isu}_n; \mathbb{R})$  is a cocycle.  $\square$

**Lemma 4.** *Via the canonical isomorphism  $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2) \simeq \mathcal{H}^3$ , the form  $\varpi_2$  is mapped to the hyperbolic volume form  $d\operatorname{vol}_{\mathcal{H}^3}$ .*

*Proof.* We use the half-space model  $\mathcal{H}^3 = \{z + tj \mid z \in \mathbb{C}, t \in \mathbb{R}, t > 0\}$ , so that the action of  $\mathrm{SL}_2(\mathbb{C})$  on  $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$  extends conformally by isometries. In particular  $\mathrm{SU}(2)$  is the stabilizer of the point  $j$ , and we use the natural map from  $\mathfrak{sl}_2$  to the tangent space  $T_j \mathcal{H}^3$  that maps  $a \in \mathfrak{sl}_2$  to the vector  $\frac{d}{dt} \exp(ta)j|_{t=0}$ . From this construction,  $\mathfrak{su}_2$  is mapped to zero and  $\mathfrak{isu}_2$  is naturally identified to tangent space to  $\mathcal{H}^3$  at  $j$ . Thus the form induced by the volume form is the result of composing a form on  $\mathfrak{isu}_2$  with the projection  $\mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{isu}_2$ . By  $\mathrm{SU}(2)$ -invariance, it suffices to check that its evaluation at an orthonormal basis is 1. The ordered basis

$$(6) \quad \left\{ \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \right\}$$

of  $\mathfrak{isu}_2$  is mapped to  $\{1, i, j\}$  via the isomorphism  $\mathfrak{isu}_2 \cong T_j \mathcal{H}^3$ , which is a positively oriented orthonormal basis, and  $\varpi$  evaluated at the basis (6) is 1.  $\square$

**Remark 2.** *The cocycle has the following precise form:*

$$\varpi = - \sum_{j < k} (ih_{jk})^\vee \wedge (ie_{jk})^\vee \wedge (if_{jk})^\vee$$

Fixing a pair of indices  $1 \leq j < k \leq n$  fixes a Lie subalgebra in  $\mathfrak{su}_n$  isomorphic to  $\mathfrak{su}_2$ . The restriction of  $\varpi$  to each of these  $\frac{n(n-1)}{2}$  copies of  $\mathfrak{su}_2$  is exactly the corresponding hyperbolic volume form.

**Remark 3.** *The imaginary part of the Cartan-Killing form,  $(x, y, z) \mapsto \Im \operatorname{tr}([x, y]z) \forall x, y, z \in \mathfrak{sl}_n$ , is cohomologous to  $-2\varpi_n$ , but it does not come from a bounded cocycle in  $\mathrm{SL}_n \mathbb{C}$  (cf. [22, Lemma 3.1] for  $n = 2$ ).*



5.1.2. *The relative part.* We now turn to the relative part of our cocycle. For this we have to understand the restriction of the form  $\varpi \in \Omega_{dR}^3(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n))$  along the canonical map  $B/T_n \rightarrow \mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$  induced by the inclusion of an arbitrary Borel subgroup  $B$ . As all Borel subgroups are conjugated in  $\mathrm{SL}_n(\mathbb{C})$  by an element of  $\mathrm{SU}(n)$ , provided by the Gram-Schmidt process, and the form  $\varpi$  is  $\mathrm{SU}(n)$ -invariant, it is enough to treat the case of our fixed Borel  $B$  of upper-triangular matrices. As we will see, because we require our trivializations to come from a bounded class there will be only one choice, and this uniqueness will then provide the coherence condition we need for computing the variation.

**Lemma 5.** *The vector space  $\Omega_{dR}^1(B/T)^B$  is generated by the closed 1-forms  $ih_s^\vee$ . In particular, the differential  $\Omega_{dR}^1(B/T)^B \rightarrow \Omega_{dR}^2(B/T)^B$  is trivial and  $H_c^1(B; \mathbb{R}) = \mathbb{R}^{n-1}$ .*

*Proof.* As before, by transitivity an element in  $\Omega_{dR}^1(B/T)^B$  is determined by its restriction to the tangent space to the identity,  $\mathfrak{b}_n/\mathfrak{h}_n$ ; i.e by a form on this tangent space invariant under the induced action by the torus  $T$ . The Borel Lie algebra  $\mathfrak{b}_n$ , the Lie algebra of the torus  $\mathfrak{h}_n$ , and the Lie algebra of strictly upper triangular matrices  $\mathfrak{ut}_n$  fit into a commutative diagram with exact row of  $T$ -modules:

$$\begin{array}{ccccccc} & & \mathfrak{ut}_n & & & & \\ & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \mathfrak{h}_n & \longrightarrow & \mathfrak{b}_n & \longrightarrow & \mathfrak{b}_n/\mathfrak{h}_n \longrightarrow 0. \end{array}$$

We view a  $T$ -invariant form on  $\mathfrak{b}_n/\mathfrak{h}_n$  as a  $T$ -invariant form  $\psi: \mathfrak{b}_n \rightarrow \mathbb{R}$  which is trivial on  $\mathfrak{h}_n$ . The action of  $T$  is readily checked to be induced by the conjugation action of  $T$  on  $B$ , hence invariance is equivalent to:

$$\forall t \in \mathfrak{h}_n, \forall b \in \mathfrak{b}_n, \quad \psi([t, b]) = 0.$$

But  $[\mathfrak{h}_n, \mathfrak{b}_n] = \mathfrak{ut}_n$ , hence  $\psi$  is in fact a form on  $\mathfrak{b}_n/\mathfrak{ut}_n$ . It is finally straightforward to check that indeed the  $n-1$  forms  $h_s^\vee$  are both closed and linearly independent.  $\square$

**Lemma 6.** *The space  $\Omega_{dR}^2(B/T)^B$  has a basis given by*

- (1) *the  $\frac{n(n-1)}{2}$  forms  $ur_{kl}^\vee \wedge ui_{kl}^\vee$  for all  $1 \leq k < l \leq n$ ;*
- (2) *the  $\frac{(n-1)(n-2)}{2}$  closed forms  $ih_s^\vee \wedge ih_r^\vee$  for all  $1 \leq s < r \leq n-1$ .*

*Proof.* Such a form, say  $\phi$ , is exactly a  $T$ -invariant and alternating 2-form on  $\mathfrak{b}_n/\mathfrak{h}_n$ . As a  $T$ -module,  $\mathfrak{b}_n/\mathfrak{h}_n = i\mathfrak{h}_n \oplus \mathfrak{ut}_n$ , hence  $\bigwedge^2 \mathfrak{b}_n/\mathfrak{h}_n = \bigwedge^2 i\mathfrak{h}_n \oplus i\mathfrak{h}_n \wedge \mathfrak{ut}_n \oplus \mathfrak{ut}_n \wedge \mathfrak{ut}_n$ . Moreover we have that  $[\mathfrak{h}_n, i\mathfrak{h}_n] = 0$  and  $[\mathfrak{h}_n, \mathfrak{ut}_n] = \mathfrak{ut}_n$ . By derivation of the invariance condition:

$$\forall a \in \mathfrak{h}_n, \forall X, Y \in \mathfrak{ut}_n, \quad \phi([a, X], Y) + \phi(X, [a, Y]) = 0.$$

From this equation one gets immediately that all forms in  $i\mathfrak{h}_n \wedge i\mathfrak{h}_n$  are invariant, and by further close inspection, that  $\phi$  on  $i\mathfrak{h}_n \wedge \mathfrak{ut}_n$  is 0.

A direct and straightforward computation shows that on  $\mathfrak{ut}_n \wedge \mathfrak{ut}_n$  the forms appearing in point (1) are the unique invariant 2-forms on this space.

Linear independence is immediate by checking on suitable elements of  $\mathfrak{b}_n/\mathfrak{h}_n$ .  $\square$

As a corollary, the trivialization we are looking for is a linear combination of the forms in Lemma 6. Let us find first a suitable candidate. Given matrices  $x, y \in \mathfrak{b}$ , write them as  $x = x_d + x_u$  and  $y = y_d + y_u$  with  $x_d, y_d \in \mathfrak{h}_n + i\mathfrak{h}_n$  diagonal and  $x_u, y_u \in \mathfrak{ut}_n$  unipotent (strictly upper triangular). Define

$$(7) \quad \begin{aligned} \beta: \mathfrak{b}_n \times \mathfrak{b}_n &\rightarrow \mathbb{R} \\ (x, y) &\mapsto \Im \mathrm{tr}(x_u {}^t \overline{y_u} - {}^t \overline{x_u} y_u)/4 = i \mathrm{tr}({}^t \overline{x_u} y_u - x_u {}^t \overline{y_u})/4. \end{aligned}$$

For  $(a_{kl}), (b_{kl}) \in \mathfrak{b}_n$  (i.e.  $a_{kl} = b_{kl} = 0$  for  $k > l$ ), (7) is equivalent to:

$$\beta((a_{kl}), (b_{kl})) = \frac{i}{4} \sum_{k < l} (\overline{a_{kl}} b_{kl} - a_{kl} \overline{b_{kl}}) = \frac{1}{2} \sum_{k < l} \Im(a_{kl} \overline{b_{kl}}),$$

so

$$\beta = \frac{1}{2} \sum_{k < l} ur_{kl} \wedge ui_{kl}.$$

In particular, in this formula coefficients in the diagonal do not occur. A straightforward computation yields:

**Lemma 7.**  $\delta(\beta) = \varpi|_{\mathfrak{b}_n}$ .

**Proposition 11.** *The form  $\beta$  above is the unique bounded 2-form  $\beta \in \Omega^2(B/T)^B$  such that  $d\beta = \varpi|_B$ . It is characterized by the fact that it is the unique trivialization that is 0 on the intersection  $B \cap B^-$ , where  $B^-$  is the opposite Borel subgroup of lower triangular matrices.*

*Proof.* Since Lemma 6 gives a basis for  $\Omega_{dR}^2(B/T)^B$ , any other invariant trivialization of  $\varpi$  restricted to  $\mathfrak{b}_n$  differs from  $\beta$  by a term of the form:

$$\sum_{s,r} \gamma_{sr} ih_s^\vee \wedge ih_r^\vee.$$

To show that the coefficients  $\gamma_{sr}$  are all 0 observe that fixing a pair of indices  $s, r$ , the exponential of the elements  $ih_s, ih_r$  give us a flat  $\mathbb{R}^2 \subset B/T$ . On this flat the volume form is trivial by direct inspection, and so are the forms  $ur_{kl}^\vee \wedge ui_{kl}^\vee$  and  $ih_p^\vee \wedge ih_q^\vee$  if  $\{p, q\} \neq \{s, r\}$ . So our invariant form on this flat restricts to the multiple  $\gamma_{sr} ih_s^\vee \wedge ih_r^\vee$  of the euclidean volume form; this is bounded if and only if  $\gamma_{sr} = 0$ .

So the unique candidate for a bounded trivialization is  $\beta$ , since we know that there has to be one bounded trivialization this is it.  $\square$

Summarizing, the class  $\text{vol}_{\mathcal{H}^3, \partial} \in H_c^3(\text{SL}_n(\mathbb{C}); \{B_i\})$  is represented in the relative de Rham complex  $\Omega_{dR}^*(\text{SL}_n(\mathbb{C})/B)^{\text{SL}_n(\mathbb{C})} \oplus \bigoplus_i \Omega_{dR}^{*-1}(B_i/B_i \cap \text{SU}(n))^{B_i}$ , by a relative cocycle where:

(1) The absolute part is given by the invariant 3-form:

$$\begin{aligned} \varpi : \bigwedge^3 \mathfrak{sl}_n & \longrightarrow \mathbb{R} \\ (A, B, C) & \longmapsto -2i \text{tr}(pr_{i\text{su}} A [pr_{i\text{su}} B, pr_{i\text{su}} C]) \end{aligned}$$

(2) The relative part is given on the copy  $\Omega^2(B_i/B_i \cap \text{SU}(n))^{B_i}$  determined by the Borel subgroup  $B_i$ , by choosing an arbitrary element  $h_i \in \text{SU}(n)$  such that  $h_i^{-1} B h_i = B_i$ , then and extending by invariance the 2-form on  $T_{\text{Id}}(B_i/B_i \cap \text{SU}(n))$  defined by  $\beta_i = \text{Ad}_H^*(\beta)$ , where:

$$\begin{aligned} \beta : \mathfrak{b}_n \times \mathfrak{b}_n & \longrightarrow \mathbb{R} \\ (x, y) & \longmapsto i \text{tr}(\overline{t_x} y_u - x_u \overline{t_y})/4. \end{aligned}$$

here  $x_u, y_u \in \mathfrak{ut}_n$  are the respective unipotent parts of  $x$  and  $y$ .

By construction the data  $(\varpi, \{\beta_i\})$  forms a relative 2-cocycle on  $\mathfrak{sl}_n$  relative to the family of Borel Lie subalgebras  $\{\mathfrak{b}_i\}$ . If we denote the associated differential forms on  $\text{SL}_n(\mathbb{C})$  and on the Borel subgroups by  $\varpi$  and  $\beta_i$ , then by de Ram's theorem, our Definition 5 of the volume of a representation takes a more precise form:

**Definition 6.** *Let  $\rho : \pi \rightarrow \text{SL}_n(\mathbb{C})$  be a representation of the fundamental group of a finite volume hyperbolic 3-manifold  $M$ . View  $M$  as the interior of a manifold with boundary  $M$ , with boundary components  $\partial M = T_1 \sqcup \dots \sqcup T_k$ . Fix a system of peripheral subgroups  $P_i$  in  $\pi$  and for each such group fix a Borel subgroup  $B_i \subset \text{SL}_n(\mathbb{C})$  such that  $\rho(P_i) \subset B_i$ . Then*

$$(8) \quad \text{Vol}(\rho) = \int_M \rho^*(\varpi) - \sum_{r=1}^k \int_{T_r} \rho^*(\beta_r).$$

5.1.3. *Volume and the Veronese embedding.* As an application let us show a formula relating the volume of a finite volume hyperbolic 3-manifold and the volume of its defining representation composed with the unique irreducible rank  $n$  representation of  $\mathrm{SL}_2(\mathbb{C})$  induced by the Veronese embedding. This formula is proved in [6, Proposition 21], with different techniques (see also [12, Thm. 1.15]).

Let  $\sigma_n: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_n(\mathbb{C})$ , denote the  $n$ -dimensional irreducible representation. Namely  $\sigma_n$  is the  $(n-1)$ -th symmetric product, induced by the Veronese embedding  $\mathbb{CP}^1 \rightarrow \mathbb{CP}^{n-1}$ .

**Proposition 12.** [6] *For  $\rho: \pi_1(M) \rightarrow \mathrm{SL}_2(\mathbb{C})$ ,  $\mathrm{vol}(\sigma_n \circ \rho) = \binom{n+1}{3} \mathrm{vol}(\rho)$ .*

Recall that given *any* family of Borel subgroups  $\{B\}$ , the map that forgets the relative part induces a natural isomorphism in continuous cohomology:

$$H_c^3(\mathrm{SL}_n(\mathbb{C}), \{B\}) \longrightarrow H_c^3(\mathrm{SL}_n(\mathbb{C})).$$

Therefore to prove Proposition 12, by the Van Est isomorphism we only need to understand the effect of the induced map  $\sigma_n: \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_n$  on the absolute part  $\varpi$  of the volume cocycle. Denote by  $\varpi_n$  this absolute part, seen as a cocycle on  $\mathfrak{sl}_n$ , to emphasize its dependence on the index  $n$ .

**Lemma 8.** *Let  $\sigma_n: \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_n$  denote the representation of Lie algebras induced by the irreducible representation that comes from the Veronese embedding. Then:*

$$\sigma_n^*(\varpi_n) = \binom{n+1}{3} \varpi_2.$$

*Proof.* The result is a consequence of the fact that  $\sigma_n(i\mathfrak{su}_2) \subset i\mathfrak{su}(n)$  and the equalities, for  $a, b \in \mathfrak{sl}_2(\mathbb{C})$ :

$$\begin{aligned} [\sigma_n(a), \sigma_n(b)] &= \sigma_n([a, b]), \\ \mathrm{tr}(\sigma_n(a)\sigma_n(b)) &= \binom{n+1}{3} \mathrm{tr}(ab). \end{aligned}$$

The first equality is just a property of Lie algebra representations. For the second one, compute the image of a basis of  $\mathfrak{sl}_2(\mathbb{C})$ :

$$\begin{aligned} \sigma_n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} n-1 & & & 0 \\ & n-3 & & \\ & & \ddots & \\ 0 & & & 1-n \end{pmatrix}, \quad \sigma_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & n-1 & & & 0 \\ & 0 & n-2 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix} \\ \text{and} \quad \sigma_n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & n-1 & 0 \end{pmatrix}. \end{aligned}$$

By bilinearity, we just need to check the formula on the basis, which is straightforward from the sums

$$\begin{aligned} (n-1)^2 + (n-3)^2 + \cdots + (1-n)^2 &= 2 \binom{n+1}{3}, \\ (n-1)1 + (n-2)2 + \cdots + 1(n-1) &= \binom{n+1}{3}. \end{aligned}$$

□

**5.2. The volume as a characteristic class.** In this section we recall briefly how a smooth deformation of a representation translates into a smooth deformation of a connection on the associated flat principal bundle. We will also recall how integration on  $M$  of pull-backs of invariant cocycles on  $\mathrm{SL}_n(\mathbb{C})$  by using a developing map give the interpretation of the volume form as a characteristic class.

Since  $\pi = \pi_1(M)$  is a discrete group, associated to our fixed representation  $\rho : \pi \longrightarrow \mathrm{SL}_n(\mathbb{C})$  there is a flat principal fibration:

$$\mathrm{SL}_n(\mathbb{C}) \hookrightarrow E_\rho \twoheadrightarrow M$$

The total space  $E_\rho$  is constructed as

$$E_\rho = \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) / \pi$$

where  $\widetilde{M}$  is the universal covering space of  $M$ ,  $\gamma \cdot (x, g) = (\gamma x, \rho(\gamma)g)$ , for  $\gamma \in \pi$ ,  $x \in \widetilde{M}$ , and  $g \in \mathrm{SL}_n(\mathbb{C})$ . The natural flat connection

$$\nabla : TE_\rho \rightarrow \mathfrak{sl}_n$$

is induced by the composition of the projection to the second factor of  $T(\widetilde{M} \times \mathrm{SL}_n(\mathbb{C})) \cong T\widetilde{M} \times T\mathrm{SL}_n(\mathbb{C})$  and the identification  $T_g\mathrm{SL}_n(\mathbb{C}) \cong \mathfrak{sl}_n$  via  $l_{g*}$  where  $l_g$  denotes left multiplication by  $g \in \mathrm{SL}_n(\mathbb{C})$ .

Notice that  $E_\rho$  is a non-compact manifold with boundary  $\partial E_\rho$  that fibres over  $\partial M$ . Recall that in Subsection 4.1 we have fixed a path from our base point in  $M$  to a base point on each boundary component. Fix a base point on each covering space of each boundary component  $\partial M_i$ , this induces commutative diagrams by sending the base point to the chosen path to  $\partial M_i$ :

$$\begin{array}{ccc} \partial \widetilde{M}_i & \hookrightarrow & \widetilde{M} \\ \downarrow & & \downarrow \\ \partial M_i & \hookrightarrow & M \end{array}$$

Since the restriction of  $\rho$  to each parabolic subgroup  $P_i \simeq \pi_1(\partial M_i)$  takes values in a Borel subgroup  $B_i$ , over the component  $\partial M_i$ , this restricted fibration

$$\mathrm{SL}_n(\mathbb{C}) \hookrightarrow \partial E_\rho \twoheadrightarrow \partial M_i$$

is obtained by extending the fibre from the flat fibration

$$B_i \hookrightarrow \partial \widetilde{M}_i \times_\rho B_i \twoheadrightarrow \partial M_i$$

along the inclusion  $B_i \hookrightarrow \mathrm{SL}_n(\mathbb{C})$ . In particular the flat connection  $\nabla$  restricted to a component  $\partial M_i$  takes values in the lie algebra  $\mathfrak{b}_i$  of the chosen Borel  $B_i$ .

As  $M$  is aspherical,  $\dim M \leq 3$ , and  $\mathrm{SL}_n(\mathbb{C})$  is 2-connected, by Whitehead's theorem there exists a  $\rho$ -equivariant map that sends the base point in  $\widetilde{M}$  to  $\mathrm{Id}$ :

$$D : \widetilde{M} \rightarrow \mathrm{SL}_n(\mathbb{C}).$$

By precomposing this map with our fixed inclusions of the universal covering spaces of the boundary components, we get for each of those a compatible developing map:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{D} & \mathrm{SL}_n(\mathbb{C}) \\ \uparrow & & \uparrow \\ \partial \widetilde{M}_i & \xrightarrow{D_i} & B_i \end{array}$$

On the one hand the developing map induces a trivialization  $\Theta_\rho$  of the flat bundle or equivalently, a section  $s_\rho$  to the fibration map:

$$\begin{array}{ccc} \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) & \longrightarrow & \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) \\ (x, g) & \longmapsto & (x, D(x)g) \\ \downarrow & & \downarrow \\ M \times \mathrm{SL}_n(\mathbb{C}) & \xrightarrow{\Theta_\rho} & E_\rho \end{array} \quad \begin{array}{ccc} \widetilde{M} & \longrightarrow & \widetilde{M} \times \mathrm{SL}_n(\mathbb{C}) \\ x & \longmapsto & (x, D(x)) \\ \downarrow & & \downarrow \\ M & \xrightarrow{s_\rho} & E_\rho \end{array}$$

Both maps are of course related:

$$s_\rho = \Theta_\rho \circ s,$$

where  $s: M \rightarrow M \times \mathrm{SL}_n(\mathbb{C})$  is the constant section of the trivial bundle, given by fixing  $\mathrm{Id} \in \mathrm{SL}_n(\mathbb{C})$  as second coordinate. The composition of the section with the flat connection

$$\nabla \circ (s_\rho)_*: TM \rightarrow \mathfrak{sl}_n$$

is used to evaluate characteristic classes of  $\mathfrak{sl}_n$ .

The trivialization  $\Theta_\rho$  is used to pull back the connection on  $E_\rho$  to the trivial bundle:

$$\nabla_\rho \stackrel{\mathrm{def}}{=} \nabla \circ (\Theta_\rho)_*: T(M \times \mathrm{SL}_n \mathbb{C}) \rightarrow \mathfrak{sl}_n$$

In this way, when we deform  $\rho$ , we deform  $\nabla_\rho$  on the trivial bundle, because

$$\nabla_\rho \circ s_* = \nabla \circ (s_\rho)_*.$$

On the other hand, the developing map models the map induced in continuous cohomology. Recall from [4, Prop. 5.4 and Cor 5.6] that if  $N$  is a smooth manifold on which  $G$  acts properly smoothly then the complex  $\Omega_{dR}^*(N)^G$  computes the continuous cohomology of  $G$ . Moreover the map in continuous cohomology induced by a continuous homomorphism  $\rho: G \rightarrow H$  can be computed by considering a  $\rho$ -equivariant map  $R: N \rightarrow M$  where  $N$  is a  $G$ -manifold as above and  $M$  an  $H$ -manifold. By definition this exactly what the developing map  $D$  is with respect to the continuous map  $\rho: \pi \rightarrow \mathrm{SL}_n(\mathbb{C})$ . Indeed, by the above cited result, we have the known fact that the canonical inclusion  $\Omega_{dR}^*(\widetilde{M})^{\pi_1(M)} \rightarrow \Omega_{dR}^*(M)$  is a quasi-isomorphism.

The same discussion holds true for each boundary component since each of these is a  $K(\mathbb{Z}^2, 1)$ , and the compatibility of the developing maps on  $M$  and on its boundary components imply that they induce via the cone construction the map:

$$H_c^*(\mathrm{SL}_n(\mathbb{C}), \{B\}) \rightarrow H^3(M, \partial M).$$

Now, since  $\mathrm{SL}_n(\mathbb{C})$  is 2-connected, the Leray-Serre spectral sequence in relative cohomology gives us a short exact sequence

$$0 \longrightarrow H^3(M, \partial M) \longrightarrow H^3(E_\rho, \partial E_\rho) \longrightarrow H^3(\mathrm{SL}_n \mathbb{C}) \longrightarrow 0$$

In particular, the volume class  $\rho^*(\mathrm{vol}_{\mathcal{H}, \partial})$  defined in Section 4 can be seen as a class in  $H^3(E_\rho, \partial E_\rho)$ . The key observation of Reznikov in [20] is that in this larger group the volume class can be interpreted as a characteristic class associated to the foliation of  $E_\rho$  induced by the flat connection.

**Proposition 13.** *Denote by  $j^*: H^3(M, \partial M) \rightarrow H^3(E_\rho, \partial E_\rho)$  the morphism induced by the projection  $E_\rho \rightarrow M$  in de Rham cohomology. Then*

$$j^*(\rho^*(\mathrm{vol}_{\mathcal{H}, \partial})) = \mathrm{Char}_{\nabla_\rho, \nabla|_{\partial M_i}}(\varpi, \{\beta_i\}).$$

*Proof.* First observe that  $\mathrm{Char}_{\nabla_\rho, \nabla|_{\partial M_i}}(\varpi, \{\beta_i\}) \in \ker(H^3(E_\rho, \partial E_\rho) \rightarrow H^3(\mathrm{SL}_n(\mathbb{C})) = \mathrm{Im} j^*)$ . Indeed, by construction, the restriction of this characteristic class to the fibre  $\mathrm{SL}_n(\mathbb{C})$  is given by the form  $\varpi$ . But the inclusion  $\mathrm{SU}(n) \rightarrow \mathrm{SL}_n(\mathbb{C})$  is a weak equivalence, hence induces an isomorphism in cohomology, and since  $\omega$  only depends on the projection on  $\mathfrak{su}_n$ , the restriction of  $\varpi$  to  $\mathrm{SU}(n)$

is the trivial form. So to check the equality we only need to show that composing with the map induced by the section  $s_\rho$  both sides of the equation agree. Recall that by construction  $(\omega, \{\beta_i\})$  is a relative cocycle that represents the hyperbolic form in  $H_c^3(\mathrm{SL}_n(\mathbb{C}), \{B\})$ . Hence by the discussion on the map  $D$  the class  $\rho * (\mathrm{vol}_{\mathcal{H}, \partial})$  is represented by the cocycle  $D^*((\omega, \{\beta_i\}))$ .

To finish the proof it is enough to show that we have a commutative diagram:

$$\begin{array}{ccccc}
 C^3(\mathfrak{sl}_n, \{\mathfrak{b}\}) & \xrightarrow{\mathrm{Char}_{\nabla_\rho, \nabla|_{\partial M_i}}} & \Omega_{dR}^3(E_\rho, \partial E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^3(M \times \mathrm{SL}_n(\mathbb{C}), \{\partial M_i \times B_i\}) \\
 \downarrow & & & \searrow s_\rho^* & \downarrow s^* \\
 \Omega_{dR}^3(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n), \{B_i/T_i\}) & \xrightarrow{\quad\quad\quad} & & & \Omega^3(M, \partial M)
 \end{array}$$

where the bottom row is induced by  $D$  and the quasi-isomorphisms  $\Omega_{dR}^*(\widetilde{M})^\pi \rightarrow \Omega_{dR}^*(M)$  and  $\Omega_{dR}^*(\widetilde{\partial M_i})^{\pi_1(\partial M_i)} \rightarrow \Omega_{dR}^*(\partial M_i)$ .

As this is a diagram on the chain level in relative cohomology, it is enough to check that the corresponding absolute maps yield commutative diagrams and are compatible, i.e. for the absolute part,

$$\begin{array}{ccccc}
 C^3(\mathfrak{sl}_n) & \xrightarrow{\mathrm{Char}_{\nabla_\rho}} & \Omega_{dR}^3(E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^3(M \times \mathrm{SL}_n(\mathbb{C})) \\
 \downarrow & & & \searrow s_\rho^* & \downarrow s^* \\
 \Omega_{dR}^3(\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)) & \xrightarrow{D^*} & \Omega^3(\widetilde{M})^\pi & \xrightarrow{\quad\quad\quad} & \Omega_{dR}^3(M)
 \end{array}$$

and analogously for the relative part:

$$\begin{array}{ccccc}
 C^2(\mathfrak{b}_i) & \xrightarrow{\mathrm{Char}_{\nabla|_{\partial M_i}}} & \Omega_{dR}^2(\partial E_\rho) & \xrightarrow{\Theta_\rho^*} & \Omega^2(\partial M_i \times B) \\
 \downarrow & & & \searrow s_\rho^* & \downarrow s^* \\
 \Omega_{dR}^2(B_i/T_i) & \xrightarrow{D^*} & \Omega^2(\widetilde{\partial M})^{\pi_1(\partial M)} & \xrightarrow{\quad\quad\quad} & \Omega_{dR}^2(\partial M)
 \end{array}$$

Both the proof commutativity of the diagrams and the compatibility are now elementary diagram chases.  $\square$

## 6. THE VARIATION FORMULA

We are now ready to collect our efforts. Recall that the group  $\mathrm{SU}(n)$  acts transitively by conjugation on the set of Borel subgroups of  $\mathrm{SL}_n(\mathbb{C})$ . Then by uniqueness of the trivialization  $\beta$  proved in Proposition 11, the trivializations of the volume form on two different Borel subgroups  $B_1$  and  $B_2$ , say  $\beta_1$  and  $\beta_2$ , are compatible in the sense of that if  $H \in \mathrm{SU}(n)$  is chosen such that  $HB_2H^{-1} = B_1$ , then for any  $b, b' \in \mathfrak{b}_2$ ,  $\beta_2(b, b') = \beta_1(\mathrm{Ad}_H b, \mathrm{Ad}_H b') = \mathrm{Ad}_H^*(\beta_1)$ .

Let  $\rho_t : \pi_1(M) \rightarrow \mathrm{SL}_n(\mathbb{C})$  be a family of representations. As we discussed before, we may think of the associated flat bundles  $E_{\rho_t}$  as being the flat bundle  $E_{\rho_0}$  but with a varying family of connections  $\nabla_t$ . The uniqueness property discussed in the previous paragraph is precisely coherence requirement of Definition 2 with respect to the subgroup  $H = \mathrm{SU}(n)$ .

We can now apply the results of Section 5.2 to compute the variation of the volume. Denote by  $H_{\mathrm{SU}(n)}^*$  the cohomology of the complexes  $C_{\mathrm{SU}(n)}^*$  defined in Section 3.2.

Then by the construction of the factorization of the variation map:

$$\begin{array}{ccccc}
 H_{\mathrm{SU}(n)}^*(\mathfrak{su}, \{\mathfrak{b}\}) & \xrightarrow{\mathrm{var}} & H_{\mathrm{SU}(n)}^{*-1}(\mathfrak{su}, \{\mathfrak{b}\}; \mathfrak{su}^\vee, \{\mathfrak{b}^\vee\}) & \xrightarrow{F} & H_{\mathrm{SU}(n)}^*(\tilde{\mathfrak{su}}, \{\tilde{\mathfrak{su}}\}) \\
 & \searrow \mathrm{Var} & & & \downarrow \mathrm{Char}_{\nabla_t, \{\tilde{\nabla}_t\}} \\
 & & & & H_{dR}^*(M, \partial M)
 \end{array}$$

we have a commutative diagram

$$\begin{array}{ccccccc}
 H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n; \mathbb{R}) & \longrightarrow & \prod H^2(\mathfrak{b}; \mathbb{R}) & \longrightarrow & H_{\mathrm{SU}(n)}^3(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathbb{R}) & \longrightarrow & H_{\mathrm{SU}(n)}^3(\mathfrak{sl}_n; \mathbb{R}) \\
 \downarrow & & \downarrow \mathrm{var} & & \downarrow \mathrm{var} & & \downarrow \mathrm{var} \\
 H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee) & \longrightarrow & \prod H^1(\mathfrak{b}; \mathfrak{b}^\vee) & \longrightarrow & H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathfrak{sl}_n^\vee, \{\mathfrak{b}^\vee\}) & \longrightarrow & H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^1(M; \mathbb{R}) & \longrightarrow & \prod H^2(\partial M) & \longrightarrow & H^3(M, \partial M) & \longrightarrow & H^3(M) \simeq 0
 \end{array}$$

Let us recall the following lemma by Cartier [9, Lemme 1]:

**Lemma 9.** *Let  $V$  be a vector space on a field  $k$  and  $A$  be a family of endomorphisms of  $V$ . Assume that  $V$  is completely reducible. Denote by  $V^\#$  the subspace of those vectors annihilated by all the  $X \in A$ . and by  $V^0$  the subspace generated by the vectors  $Xv$  ( $X \in A, v \in V$ ).*

- (1)  $V = V^\# \oplus V^0$
- (2) *If  $V$  is equipped with a differential  $d$  that commutes to the  $X \in A$ , and such that  $Xv$  is a boundary if  $v$  is a cycle, then the homology with respect to this boundary gives  $H(V) \simeq H(V^\#)$*

**Corollary 2.** *For  $* \geq 1$  and any  $n \geq 2$ ,*

$$H^*(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee) \simeq 0 \simeq H_{\mathrm{SU}(n)}^*(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee).$$

*Proof.* That  $H^*(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee) \simeq 0$  is the direct application that Cartier makes of his lemma, given that  $\mathfrak{su}_n$  is semi-simple.

For the second isomorphism, observe that as  $\mathrm{SU}(n)$  is compact. Moreover, by functoriality of the complex, its differential commutes with the action of the elements in  $\mathrm{SU}(n) - \mathrm{Id}$ . If  $d$  is a cycle, and  $X \in \mathrm{SU}(n) - \mathrm{Id}$ , then  $Xv$  is a cycle, and by acyclicity of this complex, it is a boundary. Observe that being annihilated by  $A - \mathrm{Id}$  is the same as being fixed by  $A$ , hence Lemma 9 tells us that the embedding  $C_{\mathrm{SU}(n)}^*(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee) \hookrightarrow C^*(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee)$  is a quasi-isomorphism.  $\square$

As a consequence our diagram boils down to:

$$\begin{array}{ccccccc}
 \prod H^2(\mathfrak{b}; \mathbb{R}) & \longrightarrow & H_{\mathrm{SU}(n)}^3(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathbb{R}) & \longrightarrow & H_{\mathrm{SU}(n)}^3(\mathfrak{sl}_n; \mathbb{R}) \\
 \downarrow \mathrm{var} & & \downarrow \mathrm{var} & & \downarrow \mathrm{var} \\
 0 & \longrightarrow & \prod H^1(\mathfrak{b}; \mathfrak{b}^\vee) & \longrightarrow & H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathfrak{sl}_n^\vee, \{\mathfrak{b}^\vee\}) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H^1(M; \mathbb{R}) & \longrightarrow & \prod H^2(\partial M) & \longrightarrow & H^3(M, \partial M) & \longrightarrow & 0
 \end{array}$$

In particular, the variation of the volume class is the image of a cohomology class in  $\prod H^2(\partial M)$ . To see which one we have to find an inverse to the isomorphism:

$$\prod H^1(\mathfrak{b}; \mathfrak{b}^\vee) \longrightarrow H_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathfrak{sl}_n^\vee, \{\mathfrak{b}^\vee\})$$

Unraveling the definitions it is given by the following construction. The map

$$C_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n, \{\mathfrak{b}\}; \mathfrak{sl}_n^\vee, \{\mathfrak{b}^\vee\}) \rightarrow C_{\mathrm{SU}(n)}^2(\mathfrak{sl}_n, ; \mathfrak{sl}_n^\vee, )$$

simply forgets the relative part, and acyclicity on the right hand side means that the absolute part  $\mathrm{var}(\varpi)$  of the relative cocycle  $\mathrm{var}(\varpi, \{\beta\})$  is a coboundary, say  $\mathrm{var}(\alpha) = d\gamma$ . Then the preimage of  $\mathrm{var}(\varpi, \{\beta\})$  in  $\prod H^1(\mathfrak{b}; \mathfrak{b}^\vee)$  is given by the class of the family  $\mathrm{var}(\beta) - i^*\gamma$ , where  $i^*$  is the map induced by the inclusion  $\mathfrak{b} \rightarrow \mathfrak{su}$ .

**Lemma 10.** *The image of  $\varpi$ , the absolute part of the volume cocycle, under the map  $\mathrm{var}: C_{\mathrm{SU}(n)}^3(\mathfrak{sl}_n; \mathbb{R}) \rightarrow C^2(\mathfrak{sl}_n; \mathfrak{sl}_n^\vee)$  is the coboundary of the cochain:*

$$\begin{aligned} \gamma: \mathfrak{sl}_n &\rightarrow \mathfrak{sl}_n^\vee \\ g &\mapsto h \rightsquigarrow i \operatorname{tr}(pr_{i\mathfrak{su}_n}(g)pr_{\mathfrak{su}_n}(h)) \end{aligned}$$

where  $pr_{\mathfrak{su}_n}: \mathfrak{sl}_n \rightarrow \mathfrak{su}_n$  and  $pr_{i\mathfrak{su}_n}: \mathfrak{sl}_n \rightarrow i\mathfrak{su}_n$  are the canonical projections associated to the orthogonal decomposition  $\mathfrak{sl}_n = \mathfrak{su}_n \oplus i\mathfrak{su}_n$ .

*Proof.* For  $x_1, x_2 \in \mathfrak{sl}_n$ ,

$$d(\gamma)(x_1, x_2) = x_1\gamma(x_2) - x_2\gamma(x_1) - \gamma([x_1, x_2]).$$

Recall that for  $\theta \in \mathfrak{g}^\vee$  and  $x, y \in \mathfrak{g}$ , we have  $(x\theta)(y) = -\theta([x, y])$ . Hence, for  $x_1, x_2, x_3 \in \mathfrak{sl}_n$ ,

$$\begin{aligned} d(\gamma)(x_1, x_2)(x_3) &= -\gamma(x_2)([x_1, x_3]) + \gamma(x_1)([x_2, x_3]) - \gamma([x_1, x_2])(x_3) \\ &= i \operatorname{tr}(-pr_{i\mathfrak{su}_n}(x_2)pr_{\mathfrak{su}_n}([x_1, x_3]) + pr_{i\mathfrak{su}_n}(x_1)pr_{\mathfrak{su}_n}([x_2, x_3]) - pr_{i\mathfrak{su}_n}([x_1, x_2])pr_{\mathfrak{su}_n}(x_3)) \end{aligned}$$

Since  $[\mathfrak{su}_n, \mathfrak{su}_n] \subset \mathfrak{su}_n$ ,  $[i\mathfrak{su}_n, i\mathfrak{su}_n] \subset \mathfrak{su}_n$ , and  $[i\mathfrak{su}_n, \mathfrak{su}_n] \subset i\mathfrak{su}_n$ ,

$$\begin{aligned} \delta(\gamma)(x_1, x_2)(x_3) &= i \operatorname{tr}(-pr_{i\mathfrak{su}_n}(x_2)([pr_{\mathfrak{su}_n}(x_1), pr_{\mathfrak{su}_n}(x_3)] + [pr_{i\mathfrak{su}_n}(x_1), pr_{i\mathfrak{su}_n}(x_3)]) \\ &\quad + pr_{i\mathfrak{su}_n}(x_1)([pr_{\mathfrak{su}_n}(x_2), pr_{\mathfrak{su}_n}(x_3)] + [pr_{i\mathfrak{su}_n}(x_2), pr_{i\mathfrak{su}_n}(x_3)]) \\ &\quad - ([pr_{\mathfrak{su}_n}(x_1), pr_{i\mathfrak{su}_n}(x_2)] - [pr_{i\mathfrak{su}_n}(x_1), pr_{\mathfrak{su}_n}(x_2)])pr_{\mathfrak{su}_n}(x_3)) \\ &= 2i \operatorname{tr}(pr_{i\mathfrak{su}_n}(x_1)[pr_{i\mathfrak{su}_n}(x_2), pr_{i\mathfrak{su}_n}(x_3)]) \end{aligned}$$

Here we have used that  $(A, B, C) \mapsto \operatorname{tr}(A[B, C])$  is alternated.  $\square$

Each Borel Lie algebra  $\mathfrak{b}_n$  fits into a split exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{ut}_n \longrightarrow \mathfrak{b}_n \longrightarrow \mathfrak{t}_n \longrightarrow 0$$

We have a splitting  $\mathfrak{t}_n = \mathfrak{h}_n \oplus i\mathfrak{h}_n$ . Denote by  $pr_{\mathfrak{h}_n}$  (resp.  $pr_{i\mathfrak{h}_n}$ ) the projection onto  $\mathfrak{h}_n$  (resp.  $i\mathfrak{h}_n$ ).

**Proposition 14.** *The variation of the volume of a representation is given by the sum over of the integral over each boundary component of  $\partial M$  of the image of the cohomology class of the 1-cocycle in  $C^1(\mathfrak{b}_n; \mathfrak{b}_n^\vee)$ :*

$$\begin{aligned} \zeta: \mathfrak{b}_n &\longrightarrow \mathfrak{b}_n^\vee \\ x &\longmapsto y \rightsquigarrow i \operatorname{tr}(pr_{i\mathfrak{h}_n}(x)pr_{\mathfrak{h}_n}(y)) \end{aligned}$$

under the map

$$H^1(\mathfrak{b}_n; \mathfrak{b}_n) \rightarrow H^2(\partial M).$$

*Proof.* As  $\mathrm{var}(\varpi)$  is the coboundary of  $\gamma$ , the cocycle  $(\mathrm{var}(\varpi), \{\mathrm{var}(\beta_r)\})$  is cohomologous to  $(0, \{\mathrm{var}(\beta_r) - i^*(\gamma)\})$ . Therefore, as the integral on the boundary  $\partial M$  appears subtracting in Definition 6, the variation of volume is:

$$-\sum_{r=1}^k \int_{T_r} (s_\rho^* \circ \operatorname{Char}_{\nabla_t} \circ F)(\mathrm{var}(\beta_r) - i^*(\gamma)).$$



Hence we need to prove that  $\zeta = i^*(\gamma) - \text{var}(\beta)$ . Given  $x, y \in \mathfrak{b}_n$ , write

$$x = x_d + x_u \quad \text{and} \quad y = y_d + y_u$$

with  $x_u, y_u \in \mathfrak{u}_n$  and  $x_d, y_d \in \mathfrak{h} + i\mathfrak{h}$  diagonal, their Chevalley-Jordan decomposition. Notice that  $pr_{i\mathfrak{su}_n}(x_d) = pr_{i\mathfrak{h}}(x)$  and  $pr_{\mathfrak{su}_n}(y_d) = pr_{\mathfrak{h}}(y)$  are diagonal, hence their product with elements of  $\mathfrak{ut}_n$  and  ${}^t\mathfrak{ut}_n$  have trace zero. Therefore:

$$\gamma(x)(y) = i \text{tr}(pr_{i\mathfrak{h}_n}(x)pr_{\mathfrak{h}_n}(y)) + \gamma(x_u)(y_u) = \zeta(x)(y) + \gamma(x_u)(y_u)$$

As  $pr_{i\mathfrak{su}_n}(x_u) = (x_u + {}^t\overline{x_u})/2$ ,  $pr_{\mathfrak{su}_n}(y_u) = (y_u - {}^t\overline{y_u})/2$ , and the trace vanishes on  $\mathfrak{u}_n$ ,

$$\gamma(x_u)(y_u) = i \text{tr}((x_u + {}^t\overline{x_u})(y_u - {}^t\overline{y_u}))/4 = i \text{tr}({}^t\overline{x_u}y_u - x_u {}^t\overline{y_u})/4 = \beta(x, y).$$

Hence  $i^*(\gamma) = \zeta + \text{var}(\beta)$  as claimed.  $\square$

Observe that this form we have to integrate does only depend on the projection on  $\mathfrak{b}_n/\mathfrak{ut}_n$ . Recall that corresponding to the above split exact sequence of  $\mathfrak{b}_n$  we have a split short exact sequence of Lie groups:

$$1 \longrightarrow U_n \longrightarrow B_n \longrightarrow T_n \longrightarrow 1$$

where  $U_n$  stands for the unipotent matrices, and the sequence is split by the semi-simple matrices in  $B_n$ . Then the fact that the cochain  $\zeta$  only depends on the projection onto  $\mathfrak{t}_n$  means precisely that the variation of the volume depends on the restriction of the representation  $\rho: P_i \rightarrow B_i$  only through its projection on  $B_i/U_n$ , a representation with values in an abelian group.

As an immediate corollary we have that if for each peripheral subgroup the restriction of the representation  $\rho$  take values in unipotent subgroups of  $\text{SL}_n(\mathbb{C})$  and the deformation of  $\rho$  is also boundary unipotent then the volume does not vary:

**Corollary 3.** *Representations  $\rho: \pi_1(M) \rightarrow \text{SL}_n(\mathbb{C})$  that are boundary unipotent have rigid volume.*

We now turn to a more explicit formula for the variation of the volume as encoded on each torus.

**6.1. Deforming representations on the torus.** Let  $\{\alpha, \beta\}$  be a generating set of the fundamental group of the 2-torus  $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ . They act on the universal covering  $\alpha, \beta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as the integer lattice of translations:  $\alpha(x, y) = (x + 1, y)$  and  $\beta(x, y) = (x, y + 1)$ .

By the Lie-Kolchin Theorem, the image  $\rho(\pi_1(T^2))$  is contained in a Borel subgroup  $B_n$  and up to conjugation we assume that its variation is contained in a fixed subgroup. The class we want to evaluate vanishes in  $\mathfrak{u}_n$ , so we do not need to understand the whole perturbation of  $\rho$  in  $B_n$  but just its projection to  $\pi: B_n \rightarrow B_n/U_n = \Delta_n \cong (\mathbb{C}^*)^{n-1}$ . Write

$$\pi(\rho(\alpha)) = \exp(a), \quad \pi(\rho(\beta)) = \exp(b) \in \Delta_n$$

where  $a, b \in \mathfrak{sl}_n(\mathbb{C})$  are diagonal matrices. Notice that there is an indeterminacy of the logarithm, the nontrivial entries (diagonal) of  $a$  and  $b$  are only well defined up addition of a term in to  $2\pi i\mathbb{Z}$ , but this does not affect the final result.

Since  $\Delta_n$  is abelian, for such a representation we have a  $\rho$ -equivariant map

$$\begin{aligned} D: \mathbb{R}^2 &\rightarrow B_n/U_n \\ (x, y) &\mapsto \exp(xa + yb) \end{aligned}$$

Then

$$(\nabla \circ (s_\rho)_*) \left( \frac{\partial}{\partial x} \right) = a \quad \text{and} \quad (\nabla \circ (s_\rho)_*) \left( \frac{\partial}{\partial y} \right) = b.$$

We vary the representation by varying  $a$  and  $b$ , so

$$(\dot{\nabla} \circ (s_\rho)_*) \left( \frac{\partial}{\partial x} \right) = \dot{a} \quad \text{and} \quad (\dot{\nabla} \circ (s_\rho)_*) \left( \frac{\partial}{\partial y} \right) = \dot{b}.$$

**Lemma 11.** *For  $c \in C^1(\mathfrak{g}, \mathfrak{g}^\vee)$  and a variation as above,*

$$\int_{\partial M} (s_\rho)^*(\text{Char}_{\nabla_t}(F(c))) = c(a)(\dot{b}) - c(b)(\dot{a}).$$

*Proof.* For  $Z_1, Z_2$  vector fields on  $E_\rho|_{\partial M}$ ,

$$\text{Char}_{\nabla_t}(F(c))(Z_1, Z_2) = c(\nabla(Z_1))(\dot{\nabla}(Z_2)) - c(\nabla(Z_2))(\dot{\nabla}(Z_1))$$

Setting  $Z_1 = (s_\rho)_* \left( \frac{\partial}{\partial x} \right)$  and  $Z_2 = (s_\rho)_* \left( \frac{\partial}{\partial y} \right)$ ,  $\nabla(Z_1) = a$ ,  $\dot{\nabla}(Z_1) = \dot{a}$ ,  $\nabla(Z_2) = b$ ,  $\dot{\nabla}(Z_2) = \dot{b}$ , hence

$$(s_\rho)^*(\text{Char}_{\nabla_t}(F(c))) = (c(a)(\dot{b}) - c(b)(\dot{a}))dx \wedge dy$$

As  $\int_{\partial M} dx \wedge dy = 1$ , the lemma follows.  $\square$

**Corollary 4.** *If  $a, b, \dot{a}, \dot{b} \in \mathfrak{b}_n$ , then the evaluation of the cocycle  $\zeta$  is as in Proposition 14 at the cochain in  $C^1(T^2; \mathfrak{b}_n, \mathfrak{b}'_n)$  is:*

$$\text{tr}(\Re(b)\Im(\dot{a}) - \Re(a)\Im(\dot{b})),$$

where  $\Re$  and  $\Im$  denote the usual real and imaginary part of the coefficients.

*Proof.* By Lemma 11 and Proposition 14, the evaluation of  $\zeta$  is

$$i(\text{tr}(pr_{i\mathfrak{h}_n}(a)pr_{\mathfrak{h}_n}(\dot{b})) - pr_{i\mathfrak{h}_n}(b)pr_{\mathfrak{h}_n}(\dot{a}))$$

Let  $pr_{\mathfrak{h}+i\mathfrak{h}}: \mathfrak{b}_n \rightarrow \mathfrak{h} + i\mathfrak{h}$  denote the projection to the diagonal part, then, as  $\mathfrak{h} \subset \mathfrak{su}(n)$  is the subalgebra of diagonal matrices with zero real part,

$$pr_{\mathfrak{h}} = i\Im \circ pr_{\mathfrak{h}+i\mathfrak{h}} \quad pr_{i\mathfrak{h}} = \Re \circ pr_{\mathfrak{h}+i\mathfrak{h}}$$

Thus

$$\begin{aligned} i \text{tr}(pr_{i\mathfrak{h}_n}(a)pr_{\mathfrak{h}_n}(\dot{b}) - pr_{i\mathfrak{h}_n}(b)pr_{\mathfrak{h}_n}(\dot{a})) &= i \text{tr}(\Re(a)i\Im(\dot{b}) - \Re(b)i\Im(\dot{a})) \\ &= -\text{tr}(\Re(a)\Im(\dot{b}) - \Re(b)\Im(\dot{a})) \end{aligned}$$

$\square$

**6.2. Comparison with other variation formulas.** When  $n = 2$ , write

$$a = \begin{pmatrix} \frac{l_1+i\theta_1}{2} & 0 \\ 0 & -\frac{l_1+i\theta_1}{2} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \frac{l_2+i\theta_2}{2} & 0 \\ 0 & -\frac{l_2+i\theta_2}{2} \end{pmatrix}$$

where  $\exp(a)$  is a hyperbolic isometry with translation length  $l_1$  and rotation angle  $\theta_1$ . Then, by Corollary 4, the contribution to the variation of volume is

$$\text{tr}(\Re(b)\Im(\dot{a}) - \Re(a)\Im(\dot{b})) = \frac{1}{2}(l_2\dot{\theta}_1 - l_1\dot{\theta}_2)$$

which is Hodgson's formula in [15], derived from Schläfli's formula for the variation of the volume for polyhedra in hyperbolic space.

When  $n = 3$ , the variation of volume has been discussed by Bergeron, Falbel and Guilloux in [1]. From the deformation space of decorated ideal triangulations, they obtain an equivalent formula, written differently. Firstly, their volume is 1/4 of ours, so that composing with the irreducible representation  $\sigma_3: \text{SL}_2(\mathbb{C}) \rightarrow \text{SL}_3(\mathbb{C})$  does not change the volume (in our case, by Proposition 12 it is multiplied by 4). Secondly, they work in  $\text{PGL}_3(\mathbb{C})$  instead of  $\text{SL}_3(\mathbb{C})$ , so the expression of the formula looks quite different. After taking care of these differences, their formula in Proposition 11.1.1 agrees with our formula in Corollary 4 for  $\text{SL}_3(\mathbb{C})$ . This is also noticed in Guilloux habilitation [13].

For any  $n$ , the formula on the deformation space of decorated ideal triangulations has been generalized by Dimofte, Gabella and Goncharov in [10]. A straightforward computation shows that it agrees with the formula of Theorem 1.

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